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Abstract

摘要

Anomalies are a very powerful tool in constraining theories beyond the standard model. We give a pedagogical overview of some topics illustrating the important role played by spacetime anomalies in string theory. After discussing the general problem of anomaly cancellation in quantum field theory, the focus is set on the cancellation of anomalies in type-I string theory through the Green-Schwarz mechanism. The notion of anomaly inflow is also reviewed, as well as its application to the evaluation of D-brane anomalous couplings. Finally, we briefly comment on recent developments concerning the reformulation of anomalies in the language of category theory.

反常是限制超出标准模型理论的非常有力的工具。本文对时空反常在弦论中发挥的重要作用相关部分主题做教学性概述。在讨论量子场论中反常消除的一般性问题后，我们将重点放在 I 型弦论中通过格林-施瓦茨机制实现的反常消除上。本文还回顾了反常流入的概念，及其在 D 膜反常耦合计算中的应用。最后，我们简要评述了近期用范畴论语言重新表述反常的相关进展。

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The Trouble with Anomalies

反常的问题

To the best of our knowledge, all fundamental interactions in nature are carried by quantum fields of spin one and two. Unlike scalar or spinors, the quantization of these fields requires special care in order to preserve locality and Lorentz invariance. A way to achieve this is by introducing redundant degrees of freedom living in an extended Hilbert space $\mathcal{H}$, so physical states are represented by equivalence classes under the action of a group $\mathcal{G}$. The freedom to switch from one representative to another without changing the physics is what we call gauge invariance, and the space of physical states is obtained as the quotient $\mathcal{H}_{\text{phys}} \equiv \mathcal{H}/\mathcal{G}$ [1]. Unlike standard physical symmetries mapping one state of the physical Hilbert space into another, gauge invariance only changes the state representative, the label so to speak, acting thus trivially on $\mathcal{H}_{\text{phys}}$.

据我们目前所知，自然界中所有基本相互作用都是由自旋为 1 和 2 的量子场传递的。与标量场或旋量场不同，对这些场进行量子化时需要特殊处理，才能保留定域性和洛伦兹不变性。实现这一点的一种方法是，在扩展希尔伯特空间中引入冗余自由度 $\mathcal{H}$，因此物理态由群 $\mathcal{G}$ 作用下的等价类表示。在不改变物理规律的前提下切换不同代表元的自由度就是我们所说的规范不变性，物理态空间由商空间 $\mathcal{H}_{\text{phys}} \equiv \mathcal{H}/\mathcal{G}$ 得到 [1]。与将物理希尔伯特空间中的一个态映射为另一个态的标准物理对称性不同，规范不变性仅改变态的代表元，可以说就是改变标记，因此它对 $\mathcal{H}_{\text{phys}}$ 的作用是平凡的。

For this to work it is crucial to preserve the invariance under the choice of representative. Otherwise, spurious states in $\mathcal{H}$ would enter $\mathcal{H}_{\text{phys}}$ rendering the theory nonunitary [1]. This may occur in chiral theories of the type of the standard model (SM), where gauge invariance can be broken in the process of quantization due to the necessary regularization. Whenever this happens, the theory is said to be anomalous [2] [1-3]. The requirement of chirality implies that gauge anomalies can only arise when the spacetime dimension is even.

要让这套框架成立，保留代表元选择下的不变性至关重要。否则，$\mathcal{H}$ 中的赝态会进入 $\mathcal{H}_{\text{phys}}$，导致理论不具有幺正性 [1]。这种情况可能出现在标准模型 (SM) 这类手征理论中: 由于量子化过程需要正则化，规范不变性可能被破坏。一旦发生这种情况，我们就称该理论存在反常 [2] [1-3]。手征性的要求意味着规范反常只能在偶数维时空出现。

Characterizing anomalies in gauge theories is thus of the utmost importance and its mandatory cancellation poses very strict constraints on both the theory's spectrum and interactions. The SM is a glaring example of this: the condition that all gauge and mixed gauge-gravitational anomalies cancel leads to an essentially complete determination of hypercharges, up to a global normalization. In the minimal supersymmetric standard model (MSSM), on the other hand, the introduction of an additional Higgs doublet is necessary in order to cancel the anomaly induced by the chiral higgsino (see pages 2248-2250).

因此，刻画规范理论中的反常极为重要，而反常必须抵消这一要求，对理论的能谱和相互作用都给出了非常严格的约束。标准模型就是一个十分典型的例子: 所有规范反常以及混合规范-引力反常都要抵消的条件，除整体归一化外，几乎完全确定了超荷。而在最小超对称标准模型 (MSSM) 中，必须引入额外的希格斯二重态，才能抵消手征希格斯微子诱导的反常 (参见第 2248-2250 页)。

At a diagrammatic level, anomalies are signaled by violations of the gauge Ward identities induced by the parity-violating part of one-loop diagrams with gauge current insertions and chiral fields running in the loop. In the case of the SM or their supersymmetric extensions, the relevant fields are the chiral fermions but, as we will see later, higher-dimensional theories also contain bosonic fields contributing to the anomaly. On general grounds, in $D = 2n - 2$ spacetime dimensions, the anomaly is determined by a one-loop diagram with n current insertions, which in four dimensions gives the celebrated triangle diagram. One-loop diagrams with more than n currents also contribute to the anomaly in non-Abelian theories, but their values are fully determined by the n -point diagram, so one only needs to cancel the latter.

在图论层面，反常表现为: 带规范流插入、圈中含手征场的单圈图，其宇称破缺部分导致规范 Ward 恒等式被破坏。在标准模型或其超对称扩展中，贡献反常的相关场是手征费米子，但正如我们之后会看到的，高维理论中也存在贡献反常的玻色场。一般而言，在 $D = 2n - 2$ 维时空中，反常由含 n 个流插入的单圈图确定，在四维中这就是著名的三角图。在非阿贝尔理论中，含多于 n 个流的单圈图也会对反常有贡献，但这些图的取值完全由 n 点图决定，因此只需要抵消 n 点图的反常即可。

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[1] The breakdown of gauge invariance also spoils renormalizability. From a modern effective field theory viewpoint, however, nonrenormalizability is no longer the dealbreaker it used to be.

[1] 规范不变性的破坏还会损坏可重整性。但从现代有效场论的观点来看，不可重整已经不再是过去认为的致命缺陷了。

[2] Anomalies can also affect global symmetries whose breaking does not pose any threat to the theory's consistency. Here we will not deal with these harmless anomalies.

[2] 反常也可以出现在整体对称性上，这类破缺不会威胁理论的自洽性。我们在这里不讨论这类无害的反常。

Functional methods offer a very powerful alternative to the diagrammatic analysis of anomalies. The basic object to consider is the effective action Γ_{eff} , obtained by integrating out all chiral degrees of freedom in the theory. Since these are massless fields, Γ_{eff} is a nonlocal functional resumming all one-loop diagrams with an arbitrary number of gauge current insertions.

泛函方法为反常的图论分析提供了一种非常强大的替代方案。我们需要考虑的基本对象是有效作用量 Γ_{eff} ，它由积去理论中所有手征自由度得到。由于这些都是无质量场，Γ_{eff} 是一个非局域泛函，重求和了所有带任意个规范流插入的单圈图。

Up to now, we have been talking about anomalies affecting any sort of gauge invariance, although they come in various kinds. It is however necessary to be more specific and distinguish among different types. First, we have gauge (consistent) anomalies, associated with the noninvariance of the effective action under infinitesimal gauge transformations $\delta_\chi \mathcal{A}_\mu = \partial_\mu \chi + [\mathcal{A}_\mu, \chi]$

到目前为止，我们讨论的都是影响各类规范不变性的反常，尽管反常本身分很多种。但我们有必要做更细致的讨论，区分不同类型的反常。首先，我们有规范 (相容) 反常，它和无穷小规范变换 $\delta_\chi \mathcal{A}_\mu = \partial_\mu \chi + [\mathcal{A}_\mu, \chi]$ 下有效作用量的非不变性相关。

$$\delta_\chi \Gamma_{\text{eff}} \neq 0. \tag{1}$$

Their presence leads to quantum violations of the conservation of the corresponding gauge current. For theories coupled to gravity, on the other hand, anomalies can also spoil the covariant conservation of the expectation value of the energy-momentum tensor, $\nabla_\mu \langle T^{\mu\nu} \rangle \neq 0$. These are called Einstein anomalies and are associated with the effective action's noninvariance with respect to infinitesimal diffeomorphisms

它们的存在会导致对应规范流的守恒性发生量子破缺。另一方面，对于耦合引力的理论，反常也会破坏能量动量张量期望值的协变守恒，$\nabla_\mu \langle T^{\mu\nu} \rangle \neq 0$ 。这类反常被称为爱因斯坦反常，其起源是有效作用量在无穷小微分同胚下不具有不变性

$$\delta_{\text{diff}} \Gamma_{\text{eff}} \neq 0 \,. \tag{2}$$

General relativity can also be understood as a gauge theory of local frame rotations, $e^a{}_\mu \to \Lambda^a{}_b e^b{}_\mu$, where $\Lambda^a{}_b$ is a Lorentz transformations and $e^a{}_\mu$ is the vielbein. This invariance can also be anomalous if gravity couples to chiral fields, giving rise to so-called Lorentz anomalies. They are signalled by a nonzero variation of the effective action under infinitesimal local Lorentz transformations $\Lambda = \mathbb{1} + \varepsilon$

广义相对论也可以理解为局域标架转动的规范理论，$e^a{}_\mu \to \Lambda^a{}_b e^b{}_\mu$ ，其中 $\Lambda^a{}_b$ 是洛伦兹变换，$e^a{}_\mu$ 是标架场。如果引力与手征场耦合，这种不变性也会出现反常，从而产生所谓的洛伦兹反常。其标志是，有效作用量在无穷小局域洛伦兹变换 $\Lambda = \mathbb{1} + \varepsilon$ 下发生非零变分

$$\delta_\varepsilon \Gamma_{\text{eff}} \neq 0 \,. \tag{3}$$

In fact, it is always possible to find a local counterterm that added to the effective action shifts Lorentz into Einstein anomalies and vice versa [4,5]. We will take this into account and focus our attention on Lorentz anomalies from now on.

事实上，我们总能引入一个局域抵消项，添加到有效作用量后就可以将洛伦兹反常转化为爱因斯坦反常，反之亦然 [4,5]。我们会考虑这一性质，此后将重点讨论洛伦兹反常。

In this chapter we do not enter into a detailed discussion of quantum field theory anomalies, a subject that is surveyed in a number of books and reviews (see, for instance, [2,3]). Instead, we limit our general presentation of anomalies to the basic recipes necessary to address the problem of their cancellation, with a focus on the role played by this requirement in string theory. To this aim, two particular topics are selected. First, we carry out a detailed analysis of the workings of the Green-Schwarz (GS) mechanism, which was of pivotal importance in the development of string theory and goes well beyond the ten-dimensional open superstring theory in which it was first uncovered. Next, we study the seminal notion of anomaly inflow. To illustrate its implementation, we review how anomalous D-brane (and orientifold) couplings are determined by the condition that worldvolume anomalies are cancelled by charge accretion/depletion. Our overview will be closed by a very brief discussion on the modern understanding of anomalies that has led to a suggestive connection with category theory.

本章我们不会详细讨论量子场论反常，这一主题已有大量著作与综述介绍 (例如参见 [2,3])。我们仅在概述中讲解消除反常所需的基本方法，同时重点关注这一要求在弦论中发挥的作用。为此我们选取了两个特定主题。首先，我们详细分析格林-施瓦茨 (GS) 机制的运作方式，该机制对弦论的发展至关重要，其应用范围远不止最初发现它的十维开超弦理论。接下来我们研究反常入流这一奠基性概念。为说明它的实现方式，我们回顾反常世界体积反常可以通过电荷增减抵消，从而如何确定反常 D 膜 (和定向轨形) 的耦合。最后我们会简要讨论现代对反常的理解，这一理解引出了反常与范畴论之间富有启发性的联系。

Anomaly Polynomials and Chern-Simons Forms

反常多项式与陈-西蒙斯形式

The main advantage of the effective action approach is that the anomaly can be computed from very general geometrical and topological considerations. It might be puzzling that this is possible at all, given the fact that anomalies are usually seen as the result of UV ambiguities in the calculation of the Ward-Takahashi identities. This being true, anomalies can also be interpreted as stemming from IR effects, signaled by zero-momentum poles in the expectation value of the anomalous currents themselves. It is this IR side of anomalies that makes it possible that they can be captured by studying the topological properties of the vector bundles associated with the gauge and gravitational theories [6].

有效作用量方法的主要优势在于，反常可以通过非常一般的几何与拓扑考量计算得到。反常通常被认为是 Ward-Takahashi 恒等式计算中的紫外 (UV) 歧义导致的结果，因此反常居然可以用这种方法计算，或许会令人困惑。即便上述观点没错，反常也可以被解释为红外 (IR) 效应的产物，其标志是反常流自身期望值中的零动量极点。正是反常的这一红外侧面，使得我们可以通过研究规范理论与引力理论对应向量丛的拓扑性质来得到反常 [6]。

As a general rule, anomalies on a D -dimensional curved manifold are determined by an anomaly polynomial in $D+2=2n$ dimensions [7], constructed from traces of powers of the spacetime curvature and the gauge field strength [3]

一般规律是，D 维弯曲流形上的反常由 $D+2=2n$ 维的反常多项式确定 [7]，该多项式由时空曲率和规范场强 [3] 幂次的迹构造得到。

$$I_{2n} = P\left(\operatorname{Tr}\mathcal{R}^2, \ldots, \operatorname{Tr}\mathcal{R}^n; \operatorname{tr}\mathcal{F}, \ldots, \operatorname{tr}\mathcal{F}^n\right). \tag{4}$$

Here $\mathcal{R}$ denotes the curvature two-form $\mathcal{R}^a{}_b = d\omega^a{}_b + \omega^a{}_c\omega^c{}_b$ (with $\omega^a{}_b$ the spin connection) and $\mathcal{F}$ is the gauge field strength, defined from the gauge potential $\mathcal{A}$ by $\mathcal{F} = d\mathcal{A} + \mathcal{A}^2$. In addition,"Tr" and "tr" respectively represent trace over Lorentz and gauge indices in the appropriate representation. Expression (4) shows that the anomaly polynomial is invariant under local frame rotations and gauge transformations

此处 $\mathcal{R}$ 表示曲率二形式 $\mathcal{R}^a{}_b = d\omega^a{}_b + \omega^a{}_c\omega^c{}_b$ (其中 $\omega^a{}_b$ 是自旋联络)，$\mathcal{F}$ 是规范场强，由规范势 $\mathcal{A}$ 通过 $\mathcal{F} = d\mathcal{A} + \mathcal{A}^2$ 定义。此外，"Tr"和"tr"分别代表在洛伦兹指标和规范指标对应表示下求迹。式 (4) 表明，反常多项式在局域标架转动和规范变换下不变。

$$\mathcal{R}^a{}_b \to \Lambda^a{}_c\mathcal{R}^c{}_d\left(\Lambda^T\right)^d{}_b$$

$$\mathcal{F} \to g^{-1}\mathcal{F}g \tag{5}$$

with $\Lambda^a{}_b \in \mathrm{SO}(1, D-1)$ and $g \in \mathcal{G}$, the Lorentz and gauge groups, respectively. By the index theorem, the integrated anomaly polynomial equals the index of certain differential operator in $D+2$ dimensions [5, 8, 9] .

其中 $\Lambda^a{}_b \in \mathrm{SO}(1, D-1)$ 和 $g \in \mathcal{G}$ 分别对应洛伦兹群和规范群。根据指标定理，积分后的反常多项式等于 $D+2$ 维空间中某微分算符的指标 [5, 8, 9] 。

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[3] In what follows, unless indicated otherwise, we use the conventions of Ref. [6]. Gauge fields are expressed in the language of differential forms $\mathcal{A} = \mathcal{A}_\mu dx^\mu$ and, to avoid cluttering expressions, we drop the wedge sign $\wedge$ in exterior products whenever there is no risk of ambiguity.

[3] 在下文中，除非另有说明，我们沿用文献 [6] 的约定。规范场用微分形式语言 $\mathcal{A} = \mathcal{A}_\mu dx^\mu$ 表述，为了简化表达式，在不会产生歧义的情况下，我们省略外积中的楔积符号 $\wedge$ 。

Equation (4) also shows that the anomaly polynomial I_{2n} is a closed differential form, $dI_{2n} = 0$. In fact, by Poincaré' s lemma, it is also locally exact

式 (4) 还表明，反常多项式 I_{2n} 是一个闭微分形式，$dI_{2n} = 0$ 。事实上，根据庞加莱引理，它在局域上也是恰当的。

$$I_{2n} = dI^0_{2n-1}, \tag{6}$$

where I^0_{2n-1} is the Chern-Simons form. This $(2n-1)$ -form can be integrated over a $(2n-1)$ -dimensional manifold $\mathcal{M}_{2n-1}$, whose boundary $\partial\mathcal{M}_{2n-1}$ is identified with the physical (Euclidean) spacetime. The result gives all the terms of the one-loop quantum effective action associated with the anomaly

其中 I^0_{2n-1} 是陈-西蒙斯形式。该 $(2n-1)$ 形式可以在 $(2n-1)$ 维流形 $\mathcal{M}_{2n-1}$ 上积分，该流形的边界 $\partial\mathcal{M}_{2n-1}$ 就是我们物理的 (欧氏) 时空。积分结果给出了和反常相关的单圈量子有效作用量的全部项。

$$\Gamma_{\text{eff}} = 2\pi i \int_{\mathcal{M}_{2n-1}} I^0_{2n-1}. \tag{7}$$

The global normalization of the integral is determined by either the index theorem or the diagrammatic calculation. From the point of view of the $(2n-2)$ -dimensional physical spacetime, the action (7) is nonlocal.

This is to be expected, since Γ_{eff} results from integrating out a number of massless chiral fields. Moreover, the Chern-Simons form is not uniquely determined by Eq. (6) since we are at liberty of shifting it by an arbitrary $(2n-2)$ -form, $I^0_{2n-1} \to I^0_{2n-1} + d\eta_{2n-2}$, without modifying it. This ambiguity reflects the freedom to add any local counterterm to the quantum effective action. Indeed, once integrated over $\mathcal{M}_{2n-1}$, the previous shift amounts to

积分的整体归一化由指标定理或费曼图计算确定。从 $(2n-2)$ 维物理时空的角度来看，作用量 (7) 是非局域的。这在意料之中，因为 Γ_{eff} 来源于积掉大量无质量手征场。此外，陈-西蒙斯形式并不是由式 (6) 唯一确定的，因为我们可以给它加上任意一个 $(2n-2)$ 形式 $I^0_{2n-1} \to I^0_{2n-1} + d\eta_{2n-2}$ ，而不会改变原有的反常。这种歧义反映了我们可以向量子有效作用量中任意添加局域抵消项的自由度。事实上，当我们在 $\mathcal{M}_{2n-1}$ 上积分后，上述平移相当于

$$\Gamma_{\text{eff}} \to \Gamma_{\text{eff}} + 2\pi i \int_{\partial\mathcal{M}_{2n-1}} \eta_{2n-2}, \tag{8}$$

where the last term, being an integral over physical space, is indeed local.

其中最后一项是对物理空间的积分，确实是定域的。

A crucial property of the Chern-Simons form is that, unlike the anomaly polynomial, it is not gauge or Lorentz invariant. It can be shown, however, that its gauge variation is a closed differential form, $d\delta I^0_{2n-1} = 0$, and therefore is also locally exact

陈-西蒙斯形式的一个关键性质是: 它和反常多项式不同，不具有规范不变性或洛伦兹不变性。但可以证明，它的规范变分是一个闭微分形式 $d\delta I^0_{2n-1} = 0$ ，因此在定域上也是恰当的。

$$\delta I^0_{2n-1} = dI^1_{2n-2}. \tag{9}$$

Since we are taking infinitesimal transformations, the right-hand side is linear in the gauge functions, as indicated by the superscript of I^1_{2n-2} . The anomaly is obtained by performing a gauge variation on the effective action (7)

由于我们考虑的是无穷小变换，右侧对规范函数是线性的，这也由 I^1_{2n-2} 的上标标出。反常可以通过对有效作用量 (7) 做规范变分得到。

$$\delta\Gamma_{\text{eff}} = 2\pi i \int_{\partial\mathcal{M}_{2n-1}} I^1_{2n-2}. \tag{10}$$

To be precise, this gives the so-called consistent anomaly. The name implies that this form of the anomaly satisfies the Wess-Zumino consistency conditions, a consequence of the fact that the commutator of two gauge (resp. Lorentz) transformations acting on the effective action equals the action of the transformation associated with their commutator. Although the consistent anomaly is not generically gauge covariant, a covariant form of the anomaly can be obtained by redefining the gauge current and adding counterterms [4].

准确来说，这样得到的是所谓的相容反常。这个名称表明这种形式的反常满足韦斯-祖米诺相容性条件，该条件源于: 作用在有效作用量上的两个规范 (对应洛伦兹) 变换的对易子，等于对应它们对易子的变换的作用。尽管相容反常一般不具有规范协变性，我们可以通过重新定义规范流和添加 counterterms[4] 得到反常的协变形式。

Anomaly Cancellation

反常消除

The previous discussion sets the framework to analyze anomaly cancellation: for a given theory, we should find its anomaly polynomial and see whether it is zero or not. Fortunately, there are powerful mathematical theorems at our disposal leading to a direct computation of the anomaly polynomial for any kind of fields (see [2,6]). In the case of a Weyl fermion, the Atiyah-Singer index theorem in $2n = D + 2$ dimensions leads to the explicit expression

前文的讨论已经建立了分析反常消除的框架: 对任意给定理论，我们需要求出它的反常多项式，判断它是否为零。幸运的是，我们可以借助强有力的数学定理，直接计算任意类型场的反常多项式 (参见 [2,6])。对外尔费米子，$2n = D + 2$ 维下的阿蒂亚-辛格指标定理给出了显式表达式

$$I_{\frac{1}{2}} = \left[\widehat{A}(\mathcal{M})\,\mathrm{ch}(\mathcal{F})\right]_{2n}. \tag{11}$$

The subscript on the right-hand side indicates that we retain only the $2n$ -form piece of the polynomial inside the brackets, and negative chirality spinors contribute with the same polynomial and the opposite sign. The term $\widehat{A}(\mathcal{M})$ is called the A -roof (or Dirac) genus and is determined by the curvature two-form. To give its explicit expression, we notice that $\mathcal{R}^a_b$ defines a $(2n) \times (2n)$ antisymmetric matrix of two-forms that can be brought to a block skew-diagonal form by an appropriate orthogonal transformation

右侧的下标表示我们只保留括号内多项式中的 $2n$ -形式项，负手征旋量给出相同的多项式，仅符号相反。项 $\widehat{A}(\mathcal{M})$ 被称为 A -roof(狄拉克) 亏格，由曲率二形式确定。为给出它的显式表达式，我们注意到 $\mathcal{R}^a_b$ 定义了一个由二形式构成的 $(2n) \times (2n)$ 阶反对称矩阵，可通过合适的正交变换变为分块斜对角形式

$$\frac{1}{2\pi}\mathcal{R} = \begin{pmatrix} 0 & x_1 & & & \\ -x_1 & 0 & & & \\ & & 0 & x_2 & \\ & & -x_2 & 0 & \\ & & & & \ddots \end{pmatrix}. \tag{12}$$

It is convenient to encode these skew eigenvalues in terms of the total Pontrjagin class

我们可以方便地将这些斜对角本征值用全庞特里亚金类表示

$$p(\mathcal{R}) = \prod_{i=1}^{n}\left(1 + x_i^2\right) \equiv 1 + p_1(\mathcal{R}) + p_2(\mathcal{R}) + \ldots + p_n(\mathcal{R}), \tag{13}$$

where the ℓ th Poitrjagin class is defined as the homogeneous polynomial [2, 6]

其中 ℓ 次庞特里亚金类定义为齐次多项式 [2, 6]

$$p_\ell(\mathcal{R}) = \sum_{i_1<\ldots<i_\ell}^{n} x_{i_1}^2 \ldots x_{i_\ell}^2. \tag{14}$$

These are 4ℓ -forms. Noticing that

这些都是 4ℓ -形式。注意到

$$\operatorname{Tr} \mathcal{R}^{2\ell} = (-1)^\ell \sum_{a=1}^{n} x_a^{2\ell}, \ \operatorname{Tr} \mathcal{R}^{2\ell+1} = 0, \tag{15}$$

it is easy to write $p_\ell(\mathcal{R})$ in terms of traces of powers of the curvature two-form. For the first few cases to be used later, we have

很容易将 $p_\ell(\mathcal{R})$ 用曲率二形式幂次的迹表示出来。对后续会用到的前几个情况，我们有

$$\begin{gathered} p_1 = -\frac{1}{8\pi^2} \operatorname{Tr} \mathcal{R}^2 \\ p_2 = \frac{1}{128\pi^4} \left[\left(\operatorname{Tr} \mathcal{R}^2\right)^2 - 2 \operatorname{Tr} \mathcal{R}^4 \right] \\ p_3 = -\frac{1}{3072\pi^6} \left[\left(\operatorname{Tr} \mathcal{R}^2\right)^3 - 6 \left(\operatorname{Tr} \mathcal{R}^2\right)\left(\operatorname{Tr} \mathcal{R}^4\right) + 8 \operatorname{Tr} \mathcal{R}^6 \right], \end{gathered} \tag{16}$$

where, to simplify expressions, we omit the dependence of p_k on $\mathcal{R}$. Using Pontrjagin classes, the A -roof genus in (11) can be written as the polynomial

其中为了简化表达式，我们省略了 p_k 对 $\mathcal{R}$ 的依赖关系。利用庞特里亚金类，(11) 式中的 A -roof 亏格可以写为多项式

$$\begin{aligned} \widehat{A}(\mathcal{M}) &\equiv \prod_{a=1}^{n} \frac{x_a/2}{\sinh(x_a/2)} \\ &= 1 - \frac{1}{24} p_1 + \frac{1}{5760}\left(7p_1^2 - 4p_2\right) - \frac{1}{967,680}\left(31p_1^3 - 44p_1p_2 + 16p_3\right) + \ldots . \end{aligned} \tag{17}$$

As we will soon see, Pontrjagin classes are also useful in writing the contribution of other fields to the gravitational anomaly.

我们很快就会看到，庞特里亚金类在写出其他场对引力反常的贡献时也十分有用。

The second factor on the right-hand side of Eq. (11), containing all the dependence on the gauge field, is the Chern character. It is defined by the formal series

(11) 式右侧包含所有规范场依赖关系的第二个因子是陈特征。它由形式级数定义为

$$
\mathrm{ch}(\mathcal{F}) = \sum_{j=0}^{n} \frac{1}{j!} \mathrm{tr}\left(\frac{i}{2\pi}\mathcal{F}\right)^j
$$

$$
= \mathrm{ch}_0(\mathcal{F}) + \mathrm{ch}_1(\mathcal{F}) + \ldots + \mathrm{ch}_n(\mathcal{F}), \tag{18}
$$

with the $2j$ -form

其中 $2j$ -形式

$$
\mathrm{ch}_j(\mathcal{F}) \equiv \frac{1}{j!}\mathrm{tr}_R\left(\frac{i}{2\pi}\mathcal{F}\right)^j, \tag{19}
$$

defining the j th Chern character. The subscript R indicates that the traces are taken in the representation R in which the fermions transform, with $\mathrm{ch}_0(\mathcal{F}) = N$ its dimension. Taking the exterior product of Eqs. (17) and (18) and retaining the $2n$ -form piece, we get the contribution of a chiral fermion to the anomaly polynomial in any even dimension.

定义了第 j 个陈特征。下标 R 表示迹是在费米子所属的表示 R 中计算，$\mathrm{ch}_0(\mathcal{F}) = N$ 是该表示的维数。对 (17) 式和 (18) 式取外积，只保留其中的 $2n$ -形式项，我们就得到了任意偶数维下，手征费米子对反常多项式的贡献。

Both Pontrjagin classes and Chern characters are closed forms that can be written locally in terms of the corresponding gravitational and gauge Chern-Simons forms as [4]

庞特里亚金类和陈特征都是闭形式，可以局域地用对应的引力陈-西蒙斯形式和规范陈-西蒙斯形式写为 [4]

$$
p_\ell(\mathcal{R}) = d\Omega^0_{4\ell-1}(\mathcal{R},\omega), \ \mathrm{ch}_k(\mathcal{F}) = d\omega^0_{4k-1}(\mathcal{F},\mathcal{A}). \tag{20}
$$

As explicitly indicated, these forms depend on the spin connection and the gauge field, respectively. Unlike their parent polynomials, they are not invariant under local Lorentz and gauge transformations. Their infinitesimal variations define differential forms through the descent equations

正如这里明确给出的，这些形式分别依赖自旋联络和规范场。和它们的母多项式不同，它们在定域洛伦兹变换和规范变换下不具有不变性。它们的无穷小变分可以通过降维方程得到微分形式

$$
\delta_\varepsilon \Omega^0_{4\ell-1}(\mathcal{R},\omega) = d\Omega^1_{4\ell-2}(\varepsilon,\mathcal{R},\omega),
$$

$$
\delta_\chi \omega^0_{2k-1}(\mathcal{F},\mathcal{A}) = d\omega^1_{2k-2}(\chi,\mathcal{F},\mathcal{A}). \tag{21}
$$

Further relations of this kind can be derived by considering the transformations of $\Omega^1_{2\ell-2}$ and ω^1_{2n-2} , but we will not need them here. Notice that the expressions on the right-hand side are linear in the gauge functions ε and χ , as it is explicitly indicated by the superscripts.

考虑 $\Omega^1_{2\ell-2}$ 和 ω^1_{2n-2} 的变换还可以推导出更多这类关系，但我们这里不需要用到它们。注意到正如上标明确标注的，右侧的表达式对规范函数 ε 和 χ 是线性的。

After all these mathematical preliminaries, we are ready to analyze gauge and gravitational anomalies in theories where these are brought about by Weyl fermions alone. To do so, we just need to account for all chiral fermions, add their contribution to the anomaly polynomial taking into account their chiralities and representations, and check whether the result vanishes.

完成这些数学准备后，我们现在可以分析仅由外尔费米子诱发的理论中的规范反常与引力反常了。为此，我们只需要汇总所有手征费米子，考虑它们的手征性和所属表示，把它们对反常多项式的贡献相加，然后检验结果是否为零即可。

Example I: Anomaly cancellation in the SM

例 I: 标准模型中的反常消除

As an illustration, let us study the case of the SM where all potential anomalies are sourced by spin- $\frac{1}{2}$ fermions in the fundamental representation of the gauge group. Using Eq. (11), together with the expansions (17) and (18), we can write the six-form anomaly polynomial relevant to four dimensions:

作为示例，我们来研究标准模型的情况: 在该模型中，所有潜在反常都由规范群基础表示下的自旋-$\frac{1}{2}$ 费米子产生。结合式 (11) 与展开式 (17)、(18)，我们可以写出适用于四维空间的六维反常多项式:

$$I_{\frac{1}{2}}[\mathcal{F},\mathcal{R}] = \mathrm{ch}_3 - \frac{1}{24}p_1\mathrm{ch}_1 = -\frac{i}{48\pi^3}\left[\mathrm{tr_f}\mathcal{F}^3 - \frac{1}{8}\left(\mathrm{Tr}\,\mathcal{R}^2\right)\left(\mathrm{tr_f}\mathcal{F}\right)\right], \tag{22}$$

(continued) where, for simplicity, we dropped the dependence on $\mathcal{F}$ in the Chern characters and we indicated that all traces are evaluated in the fundamental representation of $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$. They in fact can be easily computed in terms of traces over the corresponding group factors

(接上文) 为简化起见，我们省略了陈特征中对 $\mathcal{F}$ 的依赖，并注明所有迹都是在 $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ 的基础表示下计算得到。实际上这些迹可以很容易地通过对应群因子上的迹计算得到

$$\mathrm{tr_f}\mathcal{F}^3 = 2\mathrm{tr_f}\mathcal{F}_Y^3 + 3\left(\mathrm{tr_f}\mathcal{F}_Y\right)\left(\mathrm{tr_f}\mathcal{F}_L^2\right) + \mathrm{tr_f}\mathcal{F}_L^3,$$

$$\mathrm{tr_f}\mathcal{F} = 2\mathrm{tr_f}\mathcal{F}_Y \tag{23}$$

---

[4] To avoid dragging numerical factors around, our definitions of the gauge and gravitational Chern-Simons forms ω^0_{2n-1} and Ω^0_{2n-1} do include the global normalizations of the traces in the Chern characters and Pontrjagin classes. This differs from more usual conventions in the literature which define $d\omega^0_{2n-1} \equiv \mathrm{tr}\,\mathcal{F}^{2n}$.

[4] 为了省去处理数值因子的麻烦，我们对规范陈-西蒙斯形式和引力陈-西蒙斯形式 ω^0_{2n-1} 和 Ω^0_{2n-1} 的定义已经包含了陈特征和庞特里亚金类中迹的整体归一化。这与文献中定义 $d\omega^0_{2n-1} \equiv \mathrm{tr}\,\mathcal{F}^{2n}$ 的常用惯例不同。

where $\mathcal{F}_L$ and $\mathcal{F}_Y$ denote the $\mathrm{SU}(2)_L$ and $\mathrm{U}(1)_Y$ field strengths, respectively, and we have used that $\mathrm{tr_f}\mathcal{F}_L = 0$ due to the tracelessness of the generators of $\mathrm{SU}(2)_L$. We have to include in the anomaly polynomial the contributions of all Weyl fermions, weighted with the signs corresponding to their helicities. Keeping in mind that $\mathrm{SU}(2)_L$ does not couple to positive-chirality fermions, we have:

其中 $\mathcal{F}_L$ 和 $\mathcal{F}_Y$ 分别表示 $\mathrm{SU}(2)_L$ 和 $\mathrm{U}(1)_Y$ 场强，并且我们利用了 $\mathrm{tr_f}\mathcal{F}_L = 0$ ，这源于 $\mathrm{SU}(2)_L$ 生成元的无迹性。我们需要将所有外尔费米子的贡献按其螺旋度对应的符号加权后纳入反常多项式。注意到 $\mathrm{SU}(2)_L$ 不与正手征费米子耦合，因此我们得到:

$$\sum_{+}\mathrm{tr_f}\mathcal{F}_{+}^{3}-\sum_{-}\mathrm{tr_f}\mathcal{F}_{-}^{3}=2\left(\sum_{+}\mathrm{tr_f}\mathcal{F}_{Y,+}^{3}-\sum_{-}\mathrm{tr_f}\mathcal{F}_{Y,-}^{3}\right)$$
$$-3\sum_{-}\left(\mathrm{tr_f}\mathcal{F}_{Y,-}\right)\left(\mathrm{tr_f}\mathcal{F}_{L,-}^{2}\right)-\sum_{-}\mathrm{tr_f}\mathcal{F}_{L,-}^{3}. \tag{24}$$

The last term on the right-hand side of this expression vanishes because of the identity $\mathrm{tr}_f\mathcal{F}_{L,-}^3 \sim \mathrm{tr}\left(\sigma^{(i}\sigma^{j}\sigma^{k)}\right) = 0$. As for the second one, all left-handed fields couple with the same strength to the $\mathrm{SU}(2)_L$ gauge bosons so $\mathrm{tr_f}\mathcal{F}_{L,-}^2$ factors out of the sum. Summing over all leptons and (three-colored) quarks within a single family, we find the relevant traces to vanish

该表达式右侧的最后一项因恒等式 $\mathrm{tr}_f\mathcal{F}_{L,-}^3 \sim \mathrm{tr}\left(\sigma^{(i}\sigma^{j}\sigma^{k)}\right) = 0$ 而消失。至于第二项，所有左手场对 $\mathrm{SU}(2)_L$ 规范玻色子的耦合强度都相同，因此 $\mathrm{tr_f}\mathcal{F}_{L,-}^2$ 可以从求和中提出。对单个代内所有轻子和 (三色) 夸克求和后，我们发现相关迹为零

$$\sum_{+}\mathrm{tr_f}\mathcal{F}_{Y,+}^{3}-\sum_{-}\mathrm{tr_f}\mathcal{F}_{Y,-}^{3}\sim\sum_{+}Y_{+}^{3}-\sum_{-}Y_{-}^{3}=3\times2\times\left(\frac{1}{6}\right)^{3}+2\times\left(-\frac{1}{2}\right)^{3}$$
$$-3\times\left(\frac{2}{3}\right)^{3}-3\times\left(-\frac{1}{3}\right)^{3}-(-1)^{2}=0, \tag{25}$$
$$\sum_{-}\mathrm{tr_f}\mathcal{F}_{Y,-}\sim\sum_{-}Y_{-}=3\times2\times\left(\frac{1}{6}\right)+2\times\left(-\frac{1}{2}\right)=0.$$

This ensures that pure gauge anomalies cancel family by family. We are left just with the mixed gauge-gravitational term on the right-hand side of (22). Since in four dimensions gravity treats both chiralities in the same way, we only need to use the second line of (23) to find

这保证了纯规范反常逐代消除。我们只剩下式 (22) 右侧的混合规范-引力反常项。由于四维中引力对两种手征性的处理是相同的，我们只需利用式 (23) 的第二行即可得到

$$\sum_{+}\mathrm{tr_f}\mathcal{F}_{+}-\sum_{-}\mathrm{tr_f}\mathcal{F}_{-}\sim\sum_{+}Y_{+}-\sum Y_{-}=3\times2\times\left(\frac{1}{6}\right)+2\times\left(-\frac{1}{2}\right)$$
$$-3\times\left(\frac{2}{3}\right)-3\times\left(-\frac{1}{3}\right)-(-1)=0. \tag{26}$$

Thus, mixed gauge-gravitational anomalies also cancel within a single family. The SM is therefore anomaly-free.

因此，单个代内的混合规范-引力反常也得以消除。因此标准模型不存在反常。

Anomaly cancellation in the SM completely fixes the hypercharges up to a global normalization. An interesting exercise is to consider the gauge group to be $\mathrm{SU}(N_c)\times\mathrm{SU}(2)\times U(1)$ with the same family structure as in the SM but leaving the hypercharges of the fermion fields undetermined. The anomaly cancellation conditions become then a set of homogeneous equations in the hypercharges which, normalizing the hypercharge of the right-handed electron as $Y(e_R)=-1$, has a unique solution. An unphysical solution also exists when $Y(e_R)=0$.

标准模型中的反常消除将超荷完全确定到仅差一个整体归一化。一个有趣的练习是考虑规范群为 $\mathrm{SU}(N_c)\times\mathrm{SU}(2)\times U(1)$ 、费米子家族结构与标准模型相同，但费米子场的超荷未确定的情况。此时反常消除条件成为一组关于超荷的齐次方程，将右手电子的超荷归一化为 $Y(e_R)=-1$ 后，该方程组存在唯一解。当 $Y(e_R)=0$ 时，还存在一个非物理解。

The different diagrammatic contributions to the gauge anomaly can easily read from Eqs. (22) and (23). The three terms in the first line of the latter equation correspond to triangle diagrams with three hypercharge, one hypercharge and two $\mathrm{SU}(2)_L$, and three $\mathrm{SU}(2)_L$ currents, respectively. The second line, once multiplied by $\mathrm{Tr}\,\mathcal{R}^2$, gives the contribution of a triangle with one hypercharge and two graviton insertion coupling to the fermion via the energy-momentum tensor (see, for example, [1] for more details).

规范反常的不同图贡献可以很容易从式 (22) 和 (23) 中读出。后一个方程第一行的三项分别对应三个超荷、一个超荷加两个 $\mathrm{SU}(2)_L$ 、三个 $\mathrm{SU}(2)_L$ 流的三角图。第二行乘以 $\mathrm{Tr}\,\mathcal{R}^2$ 后，给出带有一个超荷插入和两个引力子插入、通过能量动量张量与费米子耦合的三角图的贡献 (更多细节参见例如文献 [1])。

As a bonus, we can work out the conditions for anomaly cancellation in the MSSM almost for free. Besides the chiral fields of the SM, whose anomalies we have seen are canceled family by family, its minimal supersymmetric extension only includes two kinds of potentially dangerous fields: the gaugi-nos and the higgsino. The first kind, transforming in the adjoint representation of the SM gauge group, do not give any nonzero contribution to the anomaly polynomial. As for the higgsino, it is an $\mathrm{SU}(2)_L$ -doublet Weyl fermion with hypercharge $\frac{1}{2}$. Its inclusion results in the nonvanishing traces

此外,我们几乎可以直接推导出 MSSM 中反常相消的条件。除了我们已经知道逐代反常相消的 SM 手征场之外, 最小超对称扩展只包含两类可能存在问题的场: 戈迪诺 (gaugino) 和希格斯 ino(higgsino)。第一类场在 SM 规范群的伴随表示下变换，对反常多项式没有任何非零贡献。而希格斯 ino 是一个 $\mathrm{SU}(2)_L$ 二重态外尔费米子，超荷为 $\frac{1}{2}$ 。纳入它会得到非零迹

$$\mathrm{tr}\,\mathcal{F}_Y^3\sim\left(\frac{1}{2}\right)^3,\ \mathrm{tr}\,\mathcal{F}_Y\sim\frac{1}{2}. \tag{27}$$

The upshot to this is that the MSSM with a single Higgs field suffers from gauge and mixed anomalies. To fix the situation, a second Higgs doublet has to be added to the one present in the SM, with an oposite chirality higgsino canceling the anomaly induced by the first one.

由此得出结论: 仅含单个希格斯场的 MSSM 存在规范反常和混合反常。为解决这一问题，需要在 SM 已有的希格斯二重态之外额外添加第二个希格斯二重态，其相反手性的希格斯 ino 会抵消第一个希格斯 ino 诱发的反常。

As explained above, diffeomorphism or local Lorentz invariance can be anomalous if parity is broken [9]. This only occurs when the dimension of the spacetime satisfies $D = 4k + 2$ (i.e., $n = 2k + 2$), with integer k . This includes the case $D = 2$ and $D = 10$, of particular interest to string theorists. When $D = 4k$ $(n = 2k + 1)$, as it is the case of our four-dimensional world, there are no pure gravitational anomalies5. This is why we did not have to care about them in our analysis of anomaly cancellation in the SM or the MSSM, although we did indeed pay attention to the gravitational contribution to the gauge anomaly.

如上所述，如果宇称破缺，微分同胚或局域洛伦兹不变性也可以存在反常 [9]。这种情况仅当时空维数满足 $D = 4k + 2$ (即 $n = 2k + 2$) 时发生，其中 k 为整数。这包括对弦论学家格外重要的 $D = 2$ 和 $D = 10$ 情形。当 $D = 4k$ $(n = 2k + 1)$ 时，也就是我们所处的四维世界，不存在纯引力反常 [5]。这就是为什么我们在分析 SM 或 MSSM 的反常相消时不需要考虑这类反常——尽管我们确实已经计入了引力对规范反常的贡献。

As in the gauge case, the gravitational anomaly in D dimensions is computed from a certain anomaly polynomial in dimension $D + 2 = 2n$ [9]. For an uncharged Weyl fermion of positive (resp. negative) helicity, it is given by the same A -roof genus defined in Eq. (17)

和规范的情况类似，D 维的引力反常由 $D + 2 = 2n$ 维的特定反常多项式计算得到 [9]。对于正 (对应负) 螺旋度的中性外尔费米子，它由式 (17) 中定义的同一个 A -亏格顶给出

$$I_{\frac{1}{2}}[\mathcal{R}] = \pm\left[\widehat{A}(\mathcal{M})\right]_{2n}. \tag{28}$$

In supergravity (SUGRA) theories, a Weyl spin- $\frac{3}{2}$ gravitino can also be the source gravitational anomalies, although being neutral it does not contribute to the gauge anomaly. Its anomaly polynomial is given by the $2n$ -form piece of [2, 5]

在超引力 (SUGRA) 理论中，自旋为 $\frac{3}{2}$ 的外尔引力微子也可以是引力反常的来源，但它作为中性粒子不对规范反常产生贡献。它的反常多项式由 [2, 5] 的 $2n$ -形式部分给出

$$\begin{aligned} I_{\frac{3}{2}}[\mathcal{R}] &\equiv \left[\prod_{a=1}^{n} \frac{x_a/2}{\sinh(x_a/2)}\right]\left(2\sum_{b=1}^{n}\cosh x_b - 1\right) \\ &= 2n - 3 + \frac{27 - 2n}{24} p_1 + \frac{1}{5760}\left[(219 + 14n)\, p_1^2 - 4(237 + 2n)\, p_2\right] \\ &+ \frac{1}{967680}\left[(597 - 62n)\, p_1^3 - 4(537 - 22n)\, p_1 p_2 + 16(507 - 2n)\, p_3\right] + \dots . \end{aligned}$$

(29)

Notice that, unlike the A -roof genus, $I_{\frac{3}{2}}$ explicitly depends on the dimension.

注意，和 A -亏格顶不同，$I_{\frac{3}{2}}$ 显式依赖于维数。

Our SM intuition might lead us to think that chiral fermions are the only fields producing gravitational anomalies and that we are done with our search of relevant polynomials to address the problem of anomaly cancellation in any dimension. This is not the case. Bosonic degrees of freedom can also play a role if they are a source of parity breaking. This is what happens if we have an r -form field B_r whose field strength is either self-dual (+) or anti-self-dual (-)

我们基于标准模型的直觉可能会认为，手征费米子是唯一产生引力反常的场，我们已经找全了处理任意维反常相消问题所需的相关多项式。事实并非如此。如果玻色自由度会导致宇称破缺，它们也可以产生贡献。当我们存在一个 r -形式场 B_r ，其场强是自对偶 (+) 或反自对偶 (-) 时，就会发生这种情况

$$dB_r = \pm \star dB_r, \tag{30}$$

where the star indicates the Hodge dual and the condition can be satisfied only for $r = (D-2)/2$. Since the star operator picks up a minus sign under parity, the

其中星号表示霍奇对偶，该条件仅在 $r = (D-2)/2$ 时可以满足。由于星算符在宇称变换下会多出一个负号，因此

---

[5] At a physical level, this happens because in $D = 4k$ CPT reverses chirality. Thus, CPT-invariant theories contain as many left-handed as right-handed fermions. Since the equivalence principle states that gravity couples to matter universally, one chirality necessarily cancels the contribution to the gravitational anomaly of the opposite one. When $D = 4k + 2$, however, CPT preserves chirality and a mismatch in the number of left- and right-handed fermons is allowed. Pure gravitational anomalies do not cancel automatically in this case.

[5] 在物理层面，该现象的成因是在 $D = 4k$ 中 CPT 反转手征性。因此，CPT 不变理论中左手征费米子与右手征费米子数量相等。根据等效原理，引力对所有物质的耦合是普适的，一种手征对引力反常的贡献必然会被相反手征的贡献抵消。然而当 $D = 4k+2$ 时，CPT 会保持手征性，允许左右手征费米子的数量不匹配，这种情况下纯引力反常无法自动抵消。

(anti)self-dual condition is not parity invariant and these fields should be taken into account when analyzing anomalies. The contribution of a self-dual tensor field to the anomaly is given in terms of the Hirzebruch polynomial $L(\mathcal{M})$ as

(反) 自对偶条件不满足宇称不变性，分析反常时必须将这些场考虑在内。自对偶张量场对反常的贡献可用希策布鲁赫多项式 $L(\mathcal{M})$ 表示为

$$I_{\rm sd}[\mathcal{R}] = -\frac{1}{8}L(\mathcal{M}) \equiv -\frac{1}{8}\prod_{a=1}^{n}\frac{x_a}{\tanh x_a} \tag{31}$$

$$= -\frac{1}{8} - \frac{1}{24}p_1 + \frac{1}{360}\left(p_1^2 - 7p_2\right) - \frac{1}{7560}\left(2p_1^3 - 13p_1p_2 + 62p_3\right) + \dots,$$

while for an anti-self-dual tensor field the polynomial has the opposite sign. Since these fields are uncharged under the gauge group, there are no terms depending on the gauge field strength.

而反自对偶张量场对应的多项式符号相反。由于这些场不携带规范群电荷，因此不存在依赖规范场强度的项。

A look at expressions (17), (29), and (31) shows that all their monomials are differential forms whose ranks are multiples of four. This means they only contain terms of rank $2n$ if $n = 2k$, or equivalently $D = 4k + 2$. This reflects the fact stated in page 2251 that pure gravitational anomalies are only possible in these dimensions.

观察式 (17)、(29) 和 (31) 可知，它们所有的单项式都是秩为 4 的倍数的微分形式。这意味着仅当 $n = 2k$ (等价于 $D = 4k + 2$) 时，式中才会包含秩为 $2n$ 的项，这也印证了第 2251 页的结论: 仅在这些维度中才可能存在纯引力反常。

Example II: Anomaly cancellation in type-IIB SUGRA

示例 II:IIB 型超引力中的反常抵消

As a second instance of anomaly cancellation, let us analyze $\mathcal{N} = 2$ ten-dimensional type-IIB SUGRA [9], the low-energy limit of type-IIB closed string theory. This is a chiral theory without gauge interactions where gravitational anomalies may arise. Its spectrum contains a number of potentially dangerous fields: two negative chirality spin- $\frac{3}{2}$ gravitini, two positive chirality spin- $\frac{1}{2}$ dilatinos, and a self-dual four-form field. All chiral fermions satisfy in addition the Majorana condition. In ten dimensions, the relevant anomaly polynomial is a 12-form that in the case of the spin- $\frac{1}{2}$ fields is read from Eqs. (17) and (28), with the result

作为反常抵消的第二个实例，我们来分析 $\mathcal{N} = 2$ 十维 IIB 型超引力 [9]，它是 IIB 闭弦理论的低能极限。这是一个没有规范相互作用的手征理论，可能会产生引力反常。它的能谱包含若干潜在危险场: 两个负手征自旋- $\frac{3}{2}$ 引力微子、两个正手征自旋- $\frac{1}{2}$ dilatino(伸缩微子)，以及一个自对偶四形式场。所有手征费米子还额外满足马约拉纳条件。在十维中，相关的反常多项式是一个 12 形式，对于自旋- $\frac{1}{2}$ 场，可由式 (17) 和 (28) 得出，结果为

$$I_{\frac{1}{2}}[\mathcal{R}] = -\frac{1}{967680}\left(31p_1^3 - 44p_1p_2 + 16p_3\right), \tag{32}$$

whereas for the gravitino [cf. (29)] we have

而对于引力微子 [参见 (29)] 我们有

$$I_{\frac{3}{2}}[\mathcal{R}] = \frac{1}{107520}\left(25p_1^3 - 180p_1p_2 + 880p_3\right). \tag{33}$$

Finally, we have the contribution of the self-dual four-form field given in Eq. (31)

最后，我们得到自对偶四形式场的贡献，由式 (31) 给出

$$I_{\mathrm{sd}}[\mathcal{R}] = -\frac{1}{7560}\left(2p_1^3 - 13p_1p_2 + 62p_3\right). \tag{34}$$

The anomaly polynomial I_{12} is given by

反常多项式 I_{12} 由下式给出

$$I_{12} = -2\times\frac{1}{2}I_{\frac{1}{2}}[\mathcal{R}] + 2\times\frac{1}{2}I_{\frac{3}{2}}[\mathcal{R}] + I_{\mathrm{sd}}[\mathcal{R}]. \tag{35}$$

The factors of 2 in front of the gravitino and dilatino contributions reflect the fact that there are two species of each kind, whereas the $\frac{1}{2}$ ' s are there because these fermions satisfy the Majorana-Weyl condition, which halves the number of real degrees of freedom with respect to a Weyl fermion. Their sign, in turn, is determined by their respective chiralities. Using the explicit expressions given above for each term, we check that

引力微子和伸缩微子贡献前的因子 2 反映了每种各有两个的事实，而 $\frac{1}{2}$ 的出现是因为这些费米子满足马约拉纳-外尔条件，该条件将实自由度的数量减半，相对于外尔费米子而言。它们的符号则由各自的手征性决定。利用上述各分项的显式表达式，我们可以验证

$$-I_{\frac{1}{2}}[\mathcal{R}] + I_{\frac{3}{2}}[\mathcal{R}] + I_{\mathrm{sd}}[\mathcal{R}] = 0. \tag{36}$$

Thus, all gravitational anomalies cancel in type-IIB SUGRA. Remember that this theory does not contain gauge fields, so we do not need to care about either gauge or mixed anomalies.

因此，IIB 型超引力中所有引力反常都被抵消了。注意该理论不包含规范场，因此我们无需考虑规范反常或混合反常。

In ten dimensions, besides $\mathcal{N} = 2$ type-IIB SUGRA, there is another interesting chiral theory: $\mathcal{N} = 1$ SUGRA that, in addition to a graviton, a dilaton, and a two-form field, also contains a left-handed gravitino and a right-handed dilatino, both of the Majorana-Weyl type. It was found in Ref. [9] that this theory is not free from gravitational anomalies. Indeed, using Eqs. (32) and (33) we see the total anomaly polynomial is nonzero

在十维中，除了 $\mathcal{N} = 2$ IIB 型超引力，还有另一个有趣的手征理论: $\mathcal{N} = 1$ 超引力，除引力子、dilaton(dilaton，dilaton) 和二形式场外，还包含一个左手引力微子和一个右手伸缩微子，二者都是马约拉纳-外尔型。文献 [9] 发现该理论无法避免引力反常。事实上，利用式 (32) 和 (33) 我们可以看到总反常多项式非零

$$\begin{aligned} I_{12} &= \frac{1}{2}I_{\frac{3}{2}}[\mathcal{R}] - \frac{1}{2}I_{\frac{1}{2}}[\mathcal{R}] \\ &= \frac{1}{15120}\left(2p_1^3 - 13p_1p_2 + 62p_3\right) \neq 0, \end{aligned} \tag{37}$$

where again the $\frac{1}{2}$ factors are due to the Majorana condition.

这里的 $\frac{1}{2}$ 因子同样源于马约拉纳条件。

This result was a source of concern given the relation of this theory to type-I superstrings, which in the 1980s were regarded as promising candidates for a unified theory of all four interactions. Its massless spectrum contains, besides a $\mathcal{N} = 1$ SUGRA multiplet, a $\mathcal{N} = 1$ super-Yang-Mills (SYM) multiplet including a gauge boson and its gaugino, a Majorana-Weyl left-handed fermion, both transforming in the adjoint representation of the gauge group. To check anomaly cancellation in type-I string theory, we add to Eq. (37) the contribution from the gaugino

这个结果曾引发担忧，因为该理论与 I 型超弦相关，I 型超弦在 1980 年代被认为是统一四种相互作用的有前景的候选理论。它的无质量能谱除包含 $\mathcal{N} = 1$ 超引力多重态外，还包含 $\mathcal{N} = 1$ 超杨-米尔斯 (SYM) 多重态，其中包括规范玻色子和它的戈金诺 (gaugino，规范微子)——马约拉纳-外尔左手费米子，二者都在规范群的伴随表示下变换。为了检验 I 型弦理论中的反常抵消，我们在式 (37) 中加入戈金诺的贡献

$$I_{12} = \frac{1}{2} I_{\frac{3}{2}} [\mathcal{R}] - \frac{1}{2} I_{\frac{1}{2}} [\mathcal{R}] + \frac{1}{2} I_{\frac{1}{2}} [\mathcal{F}, \mathcal{R}], \tag{38}$$

where last term is computed from (11)

其中最后一项由 (11) 计算得到

$$\begin{aligned} I_{\frac{1}{2}} [\mathcal{F}, \mathcal{R}] &= \mathrm{ch}_6 - \frac{1}{24} p_1 \mathrm{ch}_4 + \frac{1}{5760} \left(7p_1^2 - 4p_2\right) \mathrm{ch}_2 \\ &- \frac{N}{967680} \left(31 p_1^3 - 44 p_1 p_2 + 16 p_3\right), \end{aligned} \tag{39}$$

with all Chern characters evaluated in the adjoint representation. Putting all terms together, we get the anomaly polynomial of type-I SUGRA

所有陈特征都在伴随表示下计算。将所有项合并，我们得到 I 型超引力的反常多项式

$$\begin{aligned} 2I_{12} &= \mathrm{ch}_6 - \frac{1}{24} p_1 \mathrm{ch}_4 + \frac{1}{5760} \left(7p_1^2 - 4p_2\right) \mathrm{ch}_2 \\ &+ \frac{256 - 31N}{967680} p_1^3 + \frac{11N - 416}{241920} p_1 p_2 + \frac{496 - N}{60480} p_3, \end{aligned} \tag{40}$$

where N is the dimension of the adjoint representation of the gauge group. This nonvanishing result seems to imply that type-I string theory is anomalous and therefore should be discarded. More precisely, inspecting the anomaly polynomial we verify the existence of gauge, gravitational, and mixed anomalies. The first two are associated with hexagon diagrams with six gauge fields and six graviton fields, respectively. Mixed anomalies, on the other hand, arise from hexagon diagrams with four gauge fields and two gravitons and two gauge fields and four gravitons.

其中 N 是规范群伴随表示的维数。这个非零结果似乎暗示 I 型弦理论存在反常，因此应当被排除。更准确地说，检视反常多项式可以确认我们发现了规范反常、引力反常和混合反常。前两者分别对应六个规范场的六边形图和六个引力子的六边形图。而混合反常则来自包含四个规范场两个引力子、以及两个规范场四个引力子的六边形图。

The Green-Schwarz Solution

格林-施瓦茨解

Despite its bad prospects, in 1984 Michael Green and John Schwarz [10,11] showed that type-I string theory is anomaly-free for a particular choice of the gauge group [6] . They uncovered the existence of a nontrivial mechanism to cancel all anomalies in the theory involving the massless bosonic two-form field present in the spectrum of $\mathcal{N}=1$ SUGRA. In principle this might sound surprising, since one would not expect this field to contribute to the anomaly. Moreover, as we will see, the cancellation terms come from tree-level diagrams.

尽管 I 型弦理论前景并不乐观，迈克尔·格林与约翰·施瓦茨仍在 1984 年证明 [10,11]，当规范群取特定形式 [6] 时，I 型弦理论无反常。他们发现，该理论中存在一个非平凡机制可以抵消所有反常，该机制用到了 $\mathcal{N}=1$ 超引力谱中存在的无质量玻色二形式场。原则上这听起来令人惊讶，因为人们不会预料到该场会对反常有贡献。此外，我们将会看到，抵消项来自树图。

The GS mechanism is based on the observation that the anomaly polynomial computed in (40) can be canceled provided it admits the factorization

格林-施瓦茨机制基于以下观察: 只要 (40) 中计算得到的反常多项式可以因式分解，它就能被抵消

$$I_{12}=(\lambda \mathrm{ch}_2+p_1)X_8, \tag{41}$$

with λ a constant and X_8 an eight-form that locally can be written as $X_8=dX_7^0$. Let us discuss first how this fact leads to the cancellation of the anomaly, addressing later the problem of the conditions required for the factorization (41) to occur.

其中 λ 是常数，X_8 是一个八形式，局部可写为 $X_8=dX_7^0$ 。我们先讨论该性质如何推导出反常抵消，之后再讨论因式分解 (41) 成立需要满足的条件问题。

We know the anomaly polynomial is an exact form, $I_{12}=dI_{11}^0$, and it is possible to show that there is a solution for I_{11}^0 given by

我们知道反常多项式是一个恰当形式，$I_{12}=dI_{11}^0$ ， 可以证明 I_{11}^0 存在如下形式的解

$$I_{11}^0=\frac{1}{2}\left(\lambda\omega_3^0+\Omega_3^0\right)X_8+\frac{1}{2}\left(\lambda \mathrm{ch}_2+p_1\right)X_7^0+\frac{\alpha}{2}d\left[\left(\lambda\omega_3^0+\Omega_3^0\right)X_7^0\right], \tag{42}$$

[6] The Green-Schwarz mechanism is discussed in most books and reviews on string theory. See, for example, [12, 13] and particularly [14].

[6] 格林-施瓦茨机制在大多数弦理论教材和综述中都有讨论，例如见 [12, 13]，特别是 [14]。

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where we introduced the gravitational and gauge Chern-Simons forms Ω_3^0 and ω_3^0 defined in (20). We also included the last exact term proportional to an arbitrary coefficient α , thus exploiting the freedom of adding local counterterms to the quantum effective action. Performing a gauge transformation, we get

其中我们引入了引力陈-西蒙斯形式与规范陈-西蒙斯形式 Ω_3^0 和 ω_3^0，二者定义见 (20)。我们还添加了最后一项正比于任意系数 α 的恰当项，这利用了量子有效作用量可以添加局部 counterterm 的自由度。做规范变换后，我们得到

$$\delta I_{11}^0 = \frac{1-\alpha}{2}\left(\lambda d\omega_2^1 + d\Omega_2^1\right)X_8 + \frac{1+\alpha}{2}\left(\lambda \mathrm{ch}_2 + p_1\right)dX_6^1, \tag{43}$$

after applying the descent relation $\delta X_7^0 = dX_6^1$. Since both p_2 and ch_2 are closed forms, we see that locally $\delta I_{11}^0 = dI_{10}^1$ with

再应用下降关系 $\delta X_7^0 = dX_6^1$ 后得到。由于 p_2 和 ch_2 都是闭形式，我们可知局部有 $\delta I_{11}^0 = dI_{10}^1$，其中

$$I_{10}^1 = \frac{1-\alpha}{2}\left(\lambda\omega_2^1 + \Omega_2^1\right)X_8 + \frac{1+\alpha}{2}\left(\lambda\omega_3^0 + \Omega_3^0\right)dX_6^1. \tag{44}$$

The anomalous variation of the quantum effective action is then obtained by integrating over the ten-dimensional spacetime [cf. (10)]

量子有效作用量的反常变分可通过对十维时空积分得到 [参见 (10)]

$$\delta\Gamma_{\text{eff}} = \frac{1-\alpha}{2}c\int_{\mathcal{M}_{10}}\left(\lambda\omega_2^1 + \Omega_2^1\right)X_8 + \frac{1+\alpha}{2}c\int_{\mathcal{M}_{10}}\left(\lambda\omega_3^0 + \Omega_3^0\right)dX_6^1 \tag{45}$$

where c stands for the overall normalization of the action.

其中 c 代表作用量的整体归一化常数。

The question is whether this can be canceled by the variation of a local counterterm. It is in fact possible, provided our theory contains a two-form field B with the following gauge transformation:

问题在于反常能否被一个局部 counterterm 的变分抵消。只要我们的理论包含二形式场 B 且它满足如下规范变换，这就是可以实现的:

$$\delta C_2 = \lambda\omega_2^1 + \Omega_2^1. \tag{46}$$

When this happens, it is straightforward to check that the local counterterm

在此情形下，我们可以直接验证，局部 counterterm

$$\Gamma_{\rm ct} = -c\int_{M_{10}} C_2 X_8 - \frac{1+\alpha}{2}c\int_{M_{10}} \left(\lambda\omega_3^0 + \Omega_3^0\right) X_7^0 \tag{47}$$

has a gauge variation that exactly cancels the anomaly (45)

的规范变分恰好抵消了反常 (45)

$$\delta\Gamma_{\rm ct} = \frac{\alpha-1}{2}c\int_{\mathcal{M}_{10}} \left(\lambda\omega_2^1 + \Omega_2^1\right) X_8 - \frac{1+\alpha}{2}c\int_{\mathcal{M}_{10}} \left(\lambda\omega_3^0 + \Omega_3^0\right) dX_6^1. \tag{48}$$

At the same time, the new gauge variation of C_2 implies that its field strength has to be defined as

同时，C_2 新的规范变分意味着它的场强必须定义为

$$H = dC_2 - \lambda\omega_3^0 - \Omega_3^0, \tag{49}$$

so it remains gauge invariant.

因此场强仍然是规范不变的。

Constraints on the Gauge Group

规范群的约束

We have seen how the factorization (41) leads to the cancellation of all anomalies. What we still do not know, however, is whether this factorization can be actually achieved.

我们已经看到因式分解 (41) 如何抵消所有反常。但我们目前仍不清楚这种因式分解是否实际可行。

A look at the anomaly polynomial (40) makes it clear that the obstruction to the sought factorization lies in the presence of the sixth Chern character, ch_6 , and the third Pontrjagin class, p_3 . The latter term can be easily removed by restricting the gauge group to those whose adjoint representation has dimension $N = 496$. Doing so, the anomaly polynomial simplifies to

观察反常多项式 (40) 不难发现，目标因式分解的阻碍来自六次陈特征 ch_6 和三次庞特里亚金类 p_3 。只要将规范群限制为伴随表示维数为 $N = 496$ 的群，就可以轻松消去后一项。经过该限制后，反常多项式简化为

$$2I_{12} = \mathrm{ch}_6 - \frac{1}{24}p_1\mathrm{ch}_4 + \frac{1}{5760}\left(7p_1^2 - 4p_2\right)\mathrm{ch}_2 - \frac{1}{64}p_1^3 + \frac{1}{48}p_1p_2. \tag{50}$$

This leaves us with the problem of ch_6 . The only way to circumvent it is by further restricting the gauge group to those satisfying a relation of the type $\mathrm{ch}_6 = A\mathrm{ch}_2\mathrm{ch}_4 + B\mathrm{ch}_2^3$, for some constants A and B . Writing

X_8 as a linear combination of $\mathrm{ch}_4, \mathrm{ch}_2^2, p_1\mathrm{ch}_2, p_1^2$, and p_2 with undetermined coefficients and implementing the factorization (41), we find a unique solution where

这就给我们留下了 ch_6 的问题。规避该问题的唯一方法是进一步将规范群限制为满足 $\mathrm{ch}_6 = A\mathrm{ch}_2\mathrm{ch}_4 + B\mathrm{ch}_2^3$ 型关系的群，其中关系包含常数 A 和 B 。将 X_8 写为 $\mathrm{ch}_4, \mathrm{ch}_2^2, p_1\mathrm{ch}_2, p_1^2$ 和 p_2 的系数待定线性组合，并应用因式分解 (41)，我们得到唯一解满足:

$$\mathrm{ch}_6 = \frac{1}{720}\left(\mathrm{ch}_2\mathrm{ch}_4 - \frac{1}{1800}\mathrm{ch}_2^3\right), \tag{51}$$

$\lambda = -\frac{1}{30}$, and X_8 is given by

$\lambda = -\frac{1}{30}$ ，且 X_8 由下式给出

$$X_8 = -\frac{1}{48}\left(\mathrm{ch}_4 - \frac{1}{1800}\mathrm{ch}_2^2 - \frac{1}{60}p_1\mathrm{ch}_2 + \frac{3}{8}p_1^2 - \frac{1}{2}p_2\right). \tag{52}$$

In all previous expressions we have to keep in mind that all Chern characters are computed from traces in the adjoint representation of the gauge group.

在所有前述表达式中，我们必须记住: 所有陈特征都是根据规范群伴随表示中的迹计算得到的。

The question remains as to whether there are any groups whose adjoint representations have dimension $N = 496$ and at the same time satisfy the relation (51), which we recast as the following identity among traces:

问题仍在于，是否存在群的伴随表示维数为 $N = 496$ ，同时满足关系式 (51)，我们将该关系式改写为如下迹恒等式:

$$\mathrm{tr}_{\,\mathrm{adj}}\,\mathcal{F}^6 = \frac{1}{48}\left(\mathrm{tr}_{\,\mathrm{adj}}\,\mathcal{F}^2\right)\left(\mathrm{tr}_{\,\mathrm{adj}}\,\mathcal{F}^4\right) - \frac{1}{14,400}\left(\mathrm{tr}_{\,\mathrm{adj}}\,\mathcal{F}^2\right)^3. \tag{53}$$

The most efficient way to explore the existence of any solution to this condition is by using group theory identities relating traces of generators in the adjoint representation to those in the fundamental. We begin with the orthogonal groups $\mathrm{SO}(n)$, where we have

探索该条件是否存在解的最有效方法，是利用将伴随表示生成元迹与基础表示生成元迹联系起来的群论恒等式。我们从正交群 $\mathrm{SO}(n)$ 开始讨论，有:

$$\mathrm{tr}_{\mathrm{adj}}\mathcal{F}^6 = (n-32)\,\mathrm{tr}_{\mathrm{f}}\mathcal{F}^6 + 15\left(\mathrm{tr}_{\mathrm{f}}\mathcal{F}^2\right)\left(\mathrm{tr}_{\mathrm{f}}\mathcal{F}^4\right),$$

$$\mathrm{tr}_{\mathrm{adj}}\mathcal{F}^4 = (n-8)\,\mathrm{tr}_{\mathrm{f}}\mathcal{F}^4 + 3\left(\mathrm{tr}_{\mathrm{f}}\mathcal{F}^2\right)^2, \tag{54}$$

$$\mathrm{tr}_{\,\mathrm{adj}}\,\mathcal{F}^2 = (n-2)\,\mathrm{tr}_{\mathrm{f}}\mathcal{F}^2.$$

The key lies in the first identity. For Eq. (53) to be satisfied, it is necessary that $\mathrm{tr}_{\,\mathrm{adj}}\,\mathcal{F}^6$ be written as a product of traces in the fundamental. This only happens for SO(32), whose adjoint representation also has the right dimension $N = 496$. In fact, setting $n = 32$ in (54), we easily check that (53) is satisfied.

关键在于第一个恒等式。要满足式 (53)，$\mathrm{tr}_{\mathrm{adj}}\mathcal{F}^6$ 必须能写成基础表示下迹的乘积。该情况仅发生在 SO(32) 上，它的伴随表示也具有正确维数 $N = 496$ 。实际上，将 $n = 32$ 代入 (54)，我们可以轻松验证 (53) 成立。

Having established that the GS mechanism works for SO(32), we scan for other semisimple groups. The only one for which $\mathrm{tr}_{\mathrm{adj}}\mathcal{F}^6$ factorizes is E_8 . The adjoint and fundamental representations of this group coincide, and we have

已经证明格林-施瓦茨机制对 SO(32) 成立，我们接下来扫描其他半单群。唯一满足 $\mathrm{tr}_{\mathrm{adj}}\mathcal{F}^6$ 可因式分解的群是 E_8 。该群的伴随表示与基础表示重合，我们有:

$$\mathrm{tr}_{\mathrm{adj}}\mathcal{F}^6 = \frac{1}{7200}\left(\mathrm{tr}_{\mathrm{adj}}\mathcal{F}^2\right)^3$$

$$\mathrm{tr}_{\mathrm{adj}}\mathcal{F}^4 = \frac{1}{100}\left(\mathrm{tr}_{\mathrm{adj}}\mathcal{F}^2\right)^2 \tag{55}$$

Moreover, the adjoint representation of E_8 has dimension $N = 248$, so $E_8 \times E_8$ has the correct value of N . Using $\mathrm{tr}_{\mathrm{adj}}\mathcal{F}^n_{E_8\times E_8} = \mathrm{tr}_{\mathrm{adj}}\mathcal{F}^n_{E_8} + \mathrm{tr}_{\mathrm{adj}}\mathcal{F}'^n_{E_8}$, we confirm that (53) holds also for $E_8 \times E_8$.

此外, E_8 的伴随表示维数为 $N = 248$, 因此 $E_8 \times E_8$ 取正确值 N 。利用 $\mathrm{tr}_{\mathrm{adj}}\mathcal{F}^n_{E_8\times E_8} = \mathrm{tr}_{\mathrm{adj}}\mathcal{F}^n_{E_8} + \mathrm{tr}_{\mathrm{adj}}\mathcal{F}'^n_{E_8}$ ，我们可以确认 (53) 对 $E_8 \times E_8$ 同样成立。

Besides SO(32) and $E_8 \times E_8$, there is the somewhat trivial solution provided by the group $\mathrm{U}(1)^{496}$. In this case fermions, being in the adjoint representation, do not couple to the gauge fields, so the only contribution to the anomaly comes from the gravitational sector. Removing all terms containing Chern characters from the anomaly polynomial (50), we verify its factorization implying that anomalies are canceled by the GS mechanism. A slightly more interesting group is $\mathrm{U}(1)^{248} \times E_8$, where fermions do couple to the gauge sector through the non-Abelian factor. Taking into account that $\mathrm{tr}_{\mathrm{adj}}\mathcal{F}^{2n} = \mathrm{tr}_{\mathrm{adj}}\mathcal{F}^{2n}_{E_8}$, we use (55) to check that (53) is satisfied.

除 SO(32) 和 $E_8 \times E_8$ 外，群 $\mathrm{U}(1)^{496}$ 还给出了一个相对平凡的解。这种情况下，费米子属于伴随表示，不与规范场耦合，因此对反常的唯一贡献来自引力部分。从反常多项式 (50) 中去掉所有含陈特征的项后，我们验证了它可以因式分解，说明反常可通过格林-施瓦茨机制抵消。$\mathrm{U}(1)^{248} \times E_8$ 是一个稍显特殊的群，其费米子确实会通过非阿贝尔因子与规范 sector 耦合。考虑到 $\mathrm{tr}_{\mathrm{adj}}\mathcal{F}^{2n} = \mathrm{tr}_{\mathrm{adj}}\mathcal{F}^{2n}_{E_8}$ ，我们利用式 (55) 验证了式 (53) 成立。

With this we exhaust all possibilities. Anomalies in ten-dimensional $\mathcal{N} = 1$ SUGRA coupled to $\mathcal{N} = 1$ SYM can be canceled through the GS mechanism for just four gauge groups: SO (32) , $E_8 \times E_8$, $\mathrm{U}(1)^{496}$, and $\mathrm{U}(1)^{248} \times E_8$. Since classical type-I strings only accomodate SO $(2n)$, USp (n) , and U (n) , we conclude that SO (32) is the unique choice of the gauge group making the theory consistent. Remarkably, this is also the only group for which one-loop dilaton tadpoles cancel [7] [16].

至此我们穷尽了所有可能。对于十维 $\mathcal{N}=1$ 超引力耦合 $\mathcal{N}=1$ 超对称杨-米尔斯理论, 仅存在四个满足格林-施瓦茨抵消条件的规范群, 可通过 GS 机制抵消反常: $\mathrm{SO}\,(32)$, $E_8 \times E_8$, $\mathrm{U}(1)^{496}$ 和 $\mathrm{U}(1)^{248} \times E_8$ 。由于经典 I 型弦仅能容纳 $\mathrm{SO}\,(2n)$ 、$\mathrm{USp}\,(n)$ 和 $\mathrm{U}\,(n)$, 我们得出结论: SO (32) 是唯一能让理论自洽的规范群。值得注意的是, 它也是唯一能让单圈 dilaton 蝌蚪图抵消的群 [7] [16]。

The Physics Behind

物理机制

Let us look at the GS mechanism from a more physical angle. Besides the restriction on the gauge group just discussed, anomaly cancellation rests on the existence of a two-form field with the appropriate gauge transformation (46). The good news for type-I strings is that the theory not only allows the ”safe group” SO(32), but that it also contains a rank-two antisymmetric tensor $B_{\mu\nu}$ in the Ramond-Ramond (R-R) sector. Its kinetic term in the (string frame) low-energy effective action takes the form

让我们从更物理的角度分析格林-施瓦茨 (GS) 机制。除了刚才讨论的对规范群的限制外，反常抵消依赖于一个具有适当规范变换的二形式场的存在 (46)。I 型弦的好消息是，该理论不仅允许“安全群”SO(32)，还在拉蒙德-拉蒙德 (R-R) 扇区包含一个二阶反对称张量 $B_{\mu\nu}$ 。它在 (弦框架) 低能有效作用量中的动能项形式为

$$S_{\text{type-I}} \supset -\frac{1}{2(2\pi)^7 \alpha'^4} \int H \wedge \star H, \tag{56}$$

---

[7] There exists a subtle relation between anomalies and closed string tadpoles: spacetime anomalies in type-I string theory are linked to the presence of a nonzero tadpole for an unphysical scalar in the R-R sector. This tadpole cannot be removed through the Fischler-Susskind mechanism and only cancels if the gauge group is SO(32) [15].

[7] 反常与闭弦蝌蚪之间存在微妙关系:I 型弦理论的时空反常与 R-R 扇区非物理标量的非零蝌蚪的存在相关。这个蝌蚪无法通过菲施勒-苏斯金德机制移除，仅当规范群为 SO(32) 时才能抵消 [15]。

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where $4\pi^2\alpha' \equiv \ell_s^2$ is the string length scale. The GS mechanism can be implemented as outlined above by identifying the properly normalized two-form field C_2 above with the R-R rank two tensor, that we denote by B . Its naive field strength $H = dB$ gets modified to [8]

其中 $4\pi^2\alpha' \equiv \ell_s^2$ 是弦长标度。GS 机制可以按照上述框架实现: 将归一化正确的上述二形式场 C_2 与我们记为 B 的 R-R 二阶张量对应起来。它的裸场强 $H = dB$ 被修正为 [8]

$$H = dB - \frac{\alpha'}{4}\left[\mathrm{tr_f}\left(\mathcal{A}d\mathcal{A} + \frac{2}{3}\mathcal{A}^3\right) - \mathrm{Tr}\left(\omega d\omega + \frac{2}{3}\omega^3\right)\right], \tag{57}$$

where we used the SO(32) identity $\mathrm{tr}_{\mathrm{adj}}\mathcal{F}^2 = 30\mathrm{tr}_{\mathrm{f}}\mathcal{F}^2$ [see the last equation in (54)] to cancel the denominator of $\lambda = -\frac{1}{30}$ in (49). In addition, we need to add the GS counterterm (47), which includes the interactions

此处我们利用了 SO(32) 恒等式 $\mathrm{tr}_{\mathrm{adj}}\mathcal{F}^2 = 30\mathrm{tr}_{\mathrm{f}}\mathcal{F}^2$ [见 (54) 的最后一个方程] 抵消了 (49) 中 $\lambda = -\frac{1}{30}$ 的分母。此外，我们需要引入 GS 抵消项 (47)，其中包含相互作用项

$$S_{\mathrm{ct}} \supset \frac{1}{768\pi^5\alpha'}\int B\left[\mathrm{tr}_{\mathrm{f}}\mathcal{F}^4 - \frac{1}{8}\left(\mathrm{tr}_{\mathrm{f}}\mathcal{F}^2\right)\left(\mathrm{Tr}\,\mathcal{R}^2\right) + \frac{1}{8}\,\mathrm{Tr}\,\mathcal{R}^4 + \frac{1}{32}\left(\mathrm{Tr}\,\mathcal{R}^2\right)^2\right].$$

(58)

Here we restored the right normalization of the action and wrote again all gauge traces in the fundamental representation.

这里我们恢复了作用量的正确归一化，并将所有规范迹都重新写在了基础表示下。

These modifications to the low-energy action provide the clue to understanding how the B -field cancels the anomaly at a diagrammatic level. We know that gauge and gravitational anomalies in a ten-dimensional field theory come from hexagon diagrams with gravitons and/or gauge bosons at the vertices (see the left panel of Fig. 1). As explained in page 2254, there are contributions from all diagrams containing an even number of gravitons and gauge bosons. Now, the presence of the Chern-Simons forms ω_3^0 and Ω_3^0 in the field strength (57) induces new interactions from the kinetic term (56): vertices with a single B -field and either two gravitons or two gauge bosons. Furthermore, the part of the GS counterterm shown in (58) introduces three additional five-point vertices. They contain one B -field and either four gravitons, four gauge bosons, or two gravitons and two gauge bosons.

低能作用量的这些修正为理解 B 场如何在图论层面抵消反常提供了线索。我们知道，十维场论中的规范反常和引力反常来自顶点为引力子和/或规范玻色子的六角图 (见图 1 左幅)。正如第 2254 页所述，所有包含偶数个引力子和规范玻色子的图都有贡献。现在，场强 (57) 中陈-西蒙斯形式 ω_3^0 和 Ω_3^0 的存在，从动能项 (56) 中诱导出了新的相互作用: 包含单个 B 场与两个引力子或两个规范玻色子的顶点。此外，(58) 所示的 GS 抵消项部分引入了三个额外的五点顶点。每个顶点包含一个 B 场，以及四个引力子、四个规范玻色子，或是两个引力子加两个规范玻色子。

Due to these new interactions, there are additional diagrams contributing to the anomaly. In particular the one shown on the right of Fig. 1, where either two gravitons or two gauge bosons combine into a B -field that decays into either four gravitons, four gauge bosons, or two gravitons and two gauge bosons. The GS mechanism works because these tree-level diagrams cancel the parity-violating contribution from the hexagons, rendering the theory anomaly-free. Notice that the source of parity violation in the tree diagram is entirely in the right vertex [9] . The coupling of two gravitons or two gauge bosons and a B -field preserves parity due to the presence of the Hodge dual in the kinetic term (56).

由于这些新相互作用，存在额外的贡献反常的图，特别是图 1 右侧所示的图: 两个引力子或两个规范玻色子结合成一个 B 场，该场再衰变为四个引力子、四个规范玻色子，或是两个引力子加两个规范玻色子。GS 机制成立的原因是，这些树图抵消了六角图的宇称破坏贡献，使理论成为无反常的。注意，树图中的宇称破坏来源完全来自右顶点 [9] 。由于动能项 (56) 中存在霍奇对偶，两个引力子或两个规范玻色子与一个 B 场的耦合保持宇称。

8 The gauge transformation of the B -field by ω_2^1 is in fact also required to preserve local supersymmetry [17, 18] .

8 事实上，B 场通过 ω_2^1 的规范变换也是保持局域超对称 [17, 18] 所必需的。

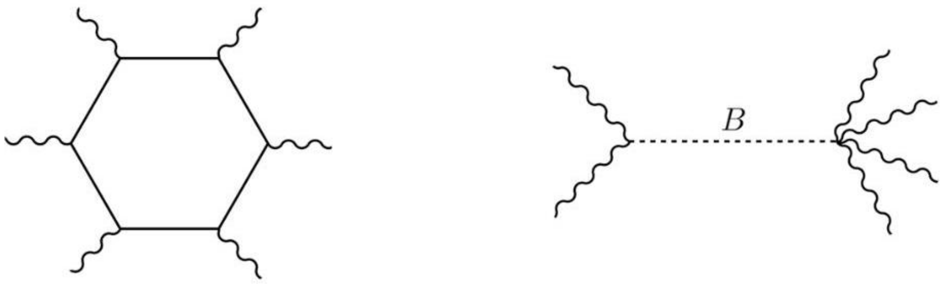

Fig. 1 Ten-dimensional anomalies are determined by a one-loop hexagon diagram as the one shown on the left panel of this figure. In the GS mechanism, their contribution to the anomaly is canceled by tree-level diagrams as the one depicted on the right panel, describing the interchange of a B -field between gravitons and/or gauge bosons. Here and in all other diagrams below, the wavy lines represent either gauge or graviton fields, as explained in the text

图 1 十维反常由如本图左幅所示的单圈六角图决定。在 GS 机制中，它们对反常的贡献被如右幅所示的树图抵消，该树图描述了引力子和/或规范玻色子之间交换一个 B 场的过程。正文中说明过，此处及以下所有图中，波浪线代表规范场或引力场

This diagrammatic interpretation brings forward the unusual character of the GS mechanism. Anomaly cancellation in quantum field theory usually proceeds by including either additional species or new couplings (or both), so the offending one-loop diagrams are canceled by other one-loop diagrams with either these extra states running in the loop or (and) new couplings showing up in the vertices. Not so in the GS mechanism. Here we have tree-level diagrams cancelling loops [10] .

这种图解诠释凸显了格林-施瓦茨机制的特殊性质。量子场论中的反常抵消通常通过引入额外粒子种类或新耦合 (或两者兼具) 实现，因此有害的单圈图会被其他单圈图抵消，这些单圈图要么在圈中包含额外粒子态，要么 (并且) 顶点中出现新耦合。格林-施瓦茨机制并非如此。在这里，树图抵消了圈图 [10] 。

To understand why this happens, we go back to a comment on page 2244 where we pointed out that anomalies can be seen as arising from IR poles in the expectation value of current, as opposed to UV ambiguities in the expectation values of the current divergence. The diagrams we are discussing contribute to the expectation value of the gauge current and the energy-momentum tensor, so anomalies are read from the residue of the parity-violating part of the hexagon diagram in the limit in which the invariant mass of the i th and j th insertions approaches zero, $s_{ij} = -2p_i \cdot p_j \to 0$. On the other hand, since the B -field is massless, its interchange in the tree diagram on the right of Fig. 1 generates a pole at zero- momentum transfer.

要理解这一现象，我们回到第 2244 页的评述，其中我们指出，反常可视为源于流期望值的红外极点，而非流散度期望值中的紫外不确定性。我们讨论的这些图对规范流和能动量张量的期望值有贡献，因此反常可从六角图宇称破坏部分的留数读出，对应极限是 i 号插入和 j 号插入的不变质量趋近于零，$s_{ij} = -2p_i \cdot p_j \to 0$ 。另一方面，由于 B 场是无质量的，它在图 1 右侧树图中的交换会在零动量转移处产生一个极点。

For the tree-level diagram to cancel the right poles from the hexagon, a number of things have to conspire. The B -field is neutral and cannot transfer gauge charge from one vertex to the other. This means that no cancellation can take place on any pole proportional to a single trace over gauge generators, a situation that we avoided by restricting the gauge group to those for which these traces factorize. Similarly, lacking Lorentz indices, it cannot cancel poles proportional to $\operatorname{Tr}\mathcal{R}^3$ either. These are the terms that we got rid of in the hexagon by setting $N = 496$.

要让树图抵消六角图的正确极点，需要多个条件共同作用。B 场是中性的，无法在两个顶点之间传递规范荷。这意味着，与单个规范生成元迹成正比的极点无法发生任何抵消，我们通过将规范群限制为满足这些迹可因子化的群已经避开了这一情况。同理，由于该场没有洛伦兹指标，它也无法抵消与 $\operatorname{Tr}\mathcal{R}^3$ 成正比的极点。这些正是我们通过设置 $N = 496$ 从六角图中消除的项。

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[9] Being an antisymmetric tensor, the coupling of the B -field to four massless bosons contains a Levi-Civita tensor that is however absent from the trivalent vertex. As a consequence, the tree-level diagram on the right of Fig. 1 violates parity.

[9] 作为反对称张量，B 场与四个无质量玻色子的耦合包含一个列维-奇维塔张量，但三顶点中不存在该张量。因此，图 1 右侧的树图破坏宇称。

[10] The closest analogy to the GS cancellation mechanism has to be found in the Wess-Zumino effective Lagrangian, where triangle anomalies are compensated by the tree-level transformation of the Nambu-Goldstone bosons [19].

[10] 与 GS 反常抵消机制最相似的例子是韦斯-祖米诺有效拉格朗日量，在该理论中，三角形反常由南部-戈德斯通玻色子的树层级变换抵消 [19]。

Type-I Superstrings and Beyond

I 型超弦及延伸理论

The modifications to the field theory action of ten-dimensional $\mathcal{N} = 1$ gauged SUGRA imposed by anomaly cancellation do not pose any problems within a field-theoretic context. Indeed, in quantum field theory we are always allowed the freedom of ”building” a Lagrangian so it fits our low energy requirements. Any couplings we might need to introduce are swept under the rug of an eventual UV completion of the theory. There is however a problem when the theory we deal with describes the low-energy dynamics of some string model. Since string theory is UV complete, both the massless states and their couplings are fully determined. Our playground is thus pretty much constrained.

反常抵消对十维 $\mathcal{N} = 1$ 定域规范超引力 (SUGRA) 场论作用量的修改在场论框架下不存在任何问题。实际上，在量子场论中我们始终拥有“构造”拉格朗日量的自由，使其满足我们的低能需求。任何我们需要引入的耦合都可以先搁置，等待理论最终完成紫外完备。但如果我们研究的理论是某个弦模型的低能动力学，问题就出现了。弦理论本身是紫外完备的，因此零质量态及其耦合都是完全确定的，我们的研究空间因此受到了相当大的约束。

This is why we need to go a step further and check whether the GS mechanism is in fact fully implemented in type-I string theory. In other words, it is not enough having the appropriate antisymmetric tensor field in the spectrum, but the new coupling stemming from the modified field strength (57) and the GS counterterm proportional to BX_8 should actually follow from the low-energy limit of the string interactions up to the last factor.

这就是为什么我们需要更进一步，检验格林-施瓦茨 (GS) 机制是否确实在 I 型弦理论中完全成立。换句话说，仅仅在谱中存在合适的反对称张量场是不够的，由修改后的场强 (57) 引出的新耦合，以及正比于 BX_8 的 GS 抵消项，都应当完全可以从弦相互作用的低能极限中自然得到。

Let us focus on pure gauge anomalies. There are three kinds of one-loop string diagrams contributing to them: the planar orientable annulus with all Yang-Mills vertex operators attached to a single boundary, the nonorientable Möbius strip with six gauge insertions on its boundary, and the nonplanar annulus with four vertices attached to one boundary and two to the other (remember that gauge bosons belong to the open string sector and therefore couple to the diagram boundaries). An explicit calculation [11] shows that for SO (32) gauge anomalies cancel among the first two topologies, whose group theory factors are in both cases single traces $\text{tr}_{\text{adj}}\,(\lambda^{a_1}\ldots\lambda^{a_6})$. As to the nonplanar diagram shown on the left of Fig. 2, it is proportional to $\text{tr}_{\text{adj}}\,(\lambda^{a_1}\lambda^{a_2})\,\text{tr}_{\text{adj}}\,(\lambda^{a_3}\ldots\lambda^{a_6})$ and does not contribute to the anomaly.

我们聚焦于纯规范反常。共有三类单圈弦图对其有贡献: 所有杨-米尔斯顶点算子都附着在单一边界上的可定向平面环面图、边界上有六个规范插入的不可定向莫比乌斯带图，以及四个顶点附着在一边界、两个顶点附着在另一边界的非平面环面图 (记住规范玻色子属于开弦部分，因此耦合到图的边界)。已有显式计算 [11] 表明，对于 SO (32)，规范反常在前两种拓扑之间抵消，二者的群论因子都是单迹 $\text{tr}_{\text{adj}}\,(\lambda^{a_1}\ldots\lambda^{a_6})$。至于图 2 左侧所示的非平面图，它正比于 $\text{tr}_{\text{adj}}\,(\lambda^{a_1}\lambda^{a_2})\,\text{tr}_{\text{adj}}\,(\lambda^{a_3}\ldots\lambda^{a_6})$，对反常没有贡献。

This proves that gauge anomalies cancel in full-fledged type-I string theory. To connect with the field-theoretic analysis of the GS mechanism, we need to look for the IR poles associated with the three diagrammatic topologies mentioned above. The two first topologies (the planar annulus and the Möbius strip) produce the poles associated to the field theory hexagon. Surprisingly, the nonplanar annulus on the left of Fig. 2 also gives a massless pole. Switching from the open to the closed string channel, this diagram is transformed into one in which the gauge bosons interact with each other through the interchange of a closed string, as shown on the right of Fig. 2. Close to the zero-momentum transfer, the amplitude is dominated by the boundary of the moduli space corresponding to a very long cylinder. Moreover, its parity-violating part is nonzero if the state running along the cylinder is the R-R rank-two antisymmetric tensor. For SO(32), the single trace in the planar annulus and Möbius strip topologies factorizes and its pole is canceled by the cylinder diagram, as required for the GS mechanism to work.

这就证明了规范反常在完整的 I 型弦理论中是抵消的。为了和 GS 机制的场论分析联系起来，我们需要寻找对应上述三种图拓扑的红外极点。前两种拓扑 (平面环面和莫比乌斯带) 产生了场论六边形图对应的极点。令人惊讶的是，图 2 左侧的非平面环面也给出了一个零质量极点。将开弦道转换为闭弦道后，这个图就变成了规范玻色子通过交换闭弦发生相互作用的形式，如图 2 右侧所示。在零动量转移附近，振幅由对应极长圆柱的模空间边界主导。此外，当沿圆柱传播的态是拉蒙-拉蒙 (R-R) 二阶反对称张量时，振幅的宇称破缺部分非零。对于 SO(32)，平面环面和莫比乌斯带拓扑中的单迹可以因子分解，其极点被圆柱图抵消，这正是 GS 机制成立所要求的结果。

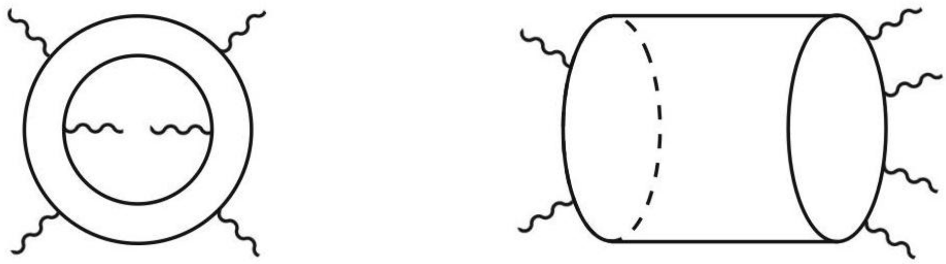

Fig. 2 In type-I string theory the contribution of the nonplanar six-point annulus diagram on the left is recast in the closed-string channel as the tree-level diagram on the right. Closed string states, among them the R-R massless B -field, propagates along the cylinder resulting in the low-energy limit in the diagram shown on the right of Fig. 1

图 2 在 I 型弦理论中，左侧非平面六点环面图的贡献可以在闭弦道中改写为右侧的树图。闭弦态 (其中包括 R-R 零质量 B 场) 沿圆柱传播，最终得到图 1 右侧所示的低能极限图。

This proves that the GS mechanism is automatically implemented in type-I string theory, at least for gauge anomalies. The analysis of gravitational and mixed anomalies requires more work. Besides the topologies already considered, now with graviton vertex operators inserted in their interior, we need to compute the contributions of the torus and the Klein bottle, both with and without a boundary. Their explicit evaluation [20] shows that all anomalies cancel at one loop. The interpretation of the tree-level diagram is now a little bit different. The relevant parity-violating pole comes again from the B -field propagating along a long cylinder. However, depending on the amplitude, there are three separate possibilities: the cylinder has two gauge boson insertions on one of its boundaries, while the other one is attached to a torus with four graviton vertex operators; it joints a sphere with two gravitons to a torus with four gravitons (cf. the diagram on the right of Fig. 3 below); or it connects a sphere with two gravitons to a torus with a boundary and two gauge bosons attached to it, while two graviton vertex operators are also inserted in its bulk.

这证明 GS 机制在 I 型弦论中自动实现，至少对规范反常而言是如此。分析引力反常和混合反常还需要更多工作。除了已经讨论过的拓扑结构 (现在其内部插入了引力子顶点算符)，我们还需要计算带边界和不带边界的环面与克莱因瓶的贡献。文献 [20] 的显式计算表明，所有反常都能在单圈水平消除。树图的诠释现在略有不同: 相关的宇称破坏极点再次来自沿长圆柱传播的 B 场。但根据振幅的不同，存在三种不同情况: 圆柱的一个边界上有两个规范玻色子插入，另一端连接着内部插入四个引力子顶点算符的环面；圆柱将一个带有两个引力子的球面连接到一个带有四个引力子的环面 (参见下图 3 右侧的图)；或是圆柱将一个带有两个引力子的球面连接到一个带边界的环面，该环面边界附着两个规范玻色子，同时圆柱本体中也插入两个引力子顶点算符。

It is difficult to exaggerate the importance of the discovery of the GS mechanism in the historical development of string theory. After the doubts sown by the results of [9] concerning the consistency of type-I string theory, the discovery of a highly nontrivial cancellation mechanism meant a very important push to the theory, so important that with it the first superstring revolution was initiated. One of the consequences was the attempts to accommodate the second ”safe” group $E_8 \times E_8$ into the framework of string theory. This led to the formulation of the heterotic string [21], the model that dominated string phenomenology until the advent of D-branes at the onset of the second superstring revolution.

GS 机制的发现对弦论的历史发展怎么强调都不为过。在文献 [9] 的结果引发了对 I 型弦论自洽性的质疑后，这种高度非平凡的反常消除机制的发现给了这一理论极为重要的推动，其影响之大直接催生了第一次超弦革命。该发现带来的其中一个结果是，人们开始尝试将第二个“安全”群 $E_8 \times E_8$ 纳入弦论的框架。这最终促成了杂化弦的表述 [21]，在第二次超弦革命开启、D 膜出现之前，这个模型一直主导着弦唯象学领域。

The low-energy dynamics of the heterotic string is also that of $\mathcal{N} = 1$ SUGRA coupled to $\mathcal{N} = 1$ SYM, although in a slightly modified fashion from the one we encountered for type-I strings [18]. This notwithstanding, the theory contains the crucial rank-two antisymmetric tensor field, this time in the NS-NS sector, and the implementation of the GS mechanism follows the same steps outlined above. The absence of an open string sector means that in the heterotic string the calculation of the anomaly only involves a single diagram, a torus with six graviton/gauge boson vertex operators as the one shown on the left of Fig. 3. The key to the cancellation of spacetime anomalies in the heterotic string lies in modular invariance [11] [23], the very same symmetry restricting the allowed gauge groups to SO (32) and $E_8 \times E_8$.

杂化弦的低能动力学同样是 $\mathcal{N} = 1$ 超引力与 $\mathcal{N} = 1$ 超对称杨-米尔斯理论耦合，只不过和我们在 I 型弦中遇到的形式相比有微小修改 [18]。尽管如此，该理论包含关键的二阶反对称张量场，在这里它属于 NS-NS sector，GS 机制的实现遵循我们上面概述的相同步骤。杂化弦没有开弦扇区，因此反常计算只涉及单个费曼图: 一个插入六个引力子/规范玻色子顶点算符的环面，如图 3 左侧所示。杂化弦时空反常能够消除的关键在于模不变性 [11] [23]，正是这个对称性将允许的规范群限定为 SO (32) 和 $E_8 \times E_8$ 。

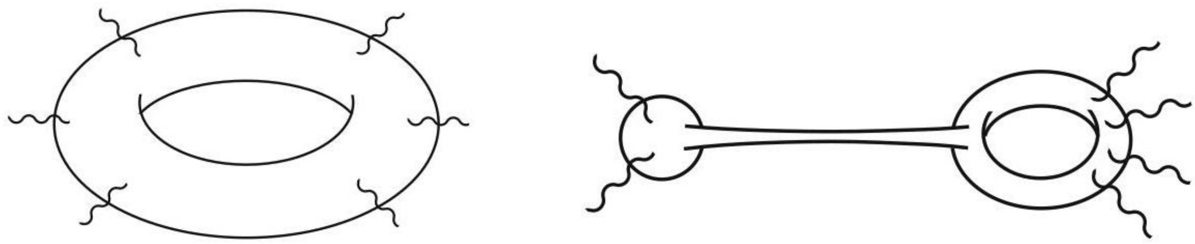

Fig. 3 All gauge and gravitational anomalies in the heterotic string stem from the torus diagram shown on the left. Low-energy poles from the field theory hexagon are canceled by the ones coming from massless closed string states running in the long propagator joining the sphere and the torus in the diagram on the right. This diagram also occurs in the calculation of the pure gravitational anomaly of type-I string theory

图 3 杂化弦中所有规范反常和引力反常都来自左侧所示的环面图。场论六角图的低能极点被右侧图中连接球面和环面的长传播子上运动的无质量闭弦态贡献的极点抵消。该图同样出现在 I 型弦论纯引力反常的计算中

Anomalies in the heterotic string are associated with contributions coming from the boundary of the moduli space of tori with six punctures. If the theory is modular invariant, this boundary has two components whose contributions cancel each other out: the limit $\tau \to i\infty$, corresponding to very long tori, and the limit in which the i th and j th punctures collide. In the first case, the diagram is dominated by massless states running along the torus and its parity-violating piece gives the pole associated with the field theory hexagon diagram. The second component corresponds to the factorization limit where the punctured torus degenerates into a sphere containing the two colliding vertex operators joined by a long cylinder to a torus with the remaining insertions (see the drawing on the right of Fig. 3). The associated parity-violating pole comes from the propagation of the B -field along the cylinder, thus implementing the GS mechanism in the low-energy field theory.

杂化弦的反常来自带六个打孔的环面模空间边界的贡献。如果理论是模不变的，该边界有两个分量，其贡献相互抵消: 极限 $\tau \to i\infty$ 对应非常长的环面，另一个极限是 i 号打孔与 j 号打孔碰撞。第一种情况，费曼图由沿环面传播的无质量态主导，其宇称破坏部分给出对应场论六角图的极点。第二个分量对应因子化极限: 带打孔的环面退化为一个包含两个碰撞顶点算符的球面，通过一个长圆柱连接到一个带有剩余顶点插入的环面 (参见图 3 右侧的示意图)。对应的宇称破坏极点来自沿圆柱传播的 B 场，由此在低能场论中实现了 GS 机制。

Some further examples of the GS mechanism

GS 机制的更多进一步实例

The GS mechanism, first identified in the context of type-I string theory, has found implementations in various scenarios. Here we discuss three particular instances of this anomaly cancellation mechanism at work.

GS 机制最初在 I 型弦论的框架中被发现，目前已在多种场景下得到应用。本文我们将讨论该反常消除机制发挥作用的三个具体实例。

The SO(16) $\times$ SO(16) nonsupersymmetric string. The SO(16) $\times$ SO(16) heterotic string [24] is a ten-dimensional nonsupersymmetric and tachyon-free fermionic model. Its spectrum, besides the universal gravity multiplet containing the graviton, dilaton, and antisymmetric rank-two tensor field, includes a positive and a negative chirality Majorana-Weyl fermions respectively transforming in the $(\mathbf{16}, \mathbf{16})$ and $(\mathbf{128}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{128})$ representations of

SO(16) $\times$ SO(16) 非超对称弦。SO(16) $\times$ SO(16) 杂化弦 [24] 是一个十维非超对称且无快子的费米模型。其谱中，除了包含引力子、dilaton、二阶反对称张量场的通用引力多重态外，还分别包含正手性和负手性的 Majorana-Weyl 费米子，它们分别属于以下表示的 $(\mathbf{16}, \mathbf{16})$ 和 $(\mathbf{128}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{128})$

(continued)

(续)

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[11] Modular invariance is also at the heart of anomaly cancellation in type-II string theory [22].

[11] 模不变性也是 II 型弦论中反常消除的核心 [22]。

$SO(16) \times SO(16)$, with16and128the fundamental and spinor representations of SO(16). Since these are the only fields contributing to the anomaly, we can write the relevant anomaly polynomial as

$SO(16) \times SO(16)$ ，其中 16 和 128 分别是 SO(16) 的基础表示和旋量表示。由于只有这些场对反常有贡献，我们可以将相关反常多项式写为

$$I_{12} = \frac{1}{2} I_{\frac{1}{2}}[\mathcal{F}, \mathcal{R}]_{(\mathbf{16},\mathbf{16})} - \frac{1}{2} I_{\frac{1}{2}}[\mathcal{F}, \mathcal{R}]_{(\mathbf{128},\mathbf{1})} - \frac{1}{2} I_{\frac{1}{2}}[\mathcal{F}, \mathcal{R}]_{(\mathbf{1},\mathbf{128})}, \tag{59}$$

where once more the $\frac{1}{2}$ factors account for the number of degrees of freedom of a Majorana-Weyl spinor and the subscripts remind us what representation we should use to evaluate the traces in the corresponding Chern characters. It is a peculiarity of this model that, unlike the type-I superstring, we have fermions transforming in various representations of the gauge group. Interestingly, looking at Eq. (39) we see that all pure gravitational anomaly terms cancel among each other, due to the identity $16^2 - 128 - 128 = 0$.

其中 $\frac{1}{2}$ 因子再次说明了 Majorana-Weyl 旋子自由度的数量，下标也提醒我们在对应陈特征中计算迹时应当采用哪个表示。该模型的一个特殊之处在于，和 I 型超弦不同，我们存在属于规范群不同表示的费米子。有趣的是，观察式 (39) 可以发现，由于恒等式 $16^2 - 128 - 128 = 0$ ，所有纯引力反常项相互抵消。

We are left then with pure and mixed gauge anomalies. In order to check their cancellation it is convenient to write all Chern characters using traces in the fundamental representation 16 of $SO(16)$. Remarkably, when this is done [24] we find the factorized result

剩下的就是纯规范反常和混合规范反常。为了验证这些反常的消除，方便起见我们用 $SO(16)$ 基础表示 16 中的迹写出所有陈特征。令人惊讶的是，完成后我们得到了因式分解的结果 [24]

$$I_{12} = \left(\mathrm{ch}_2 + \mathrm{ch}_2' - p_1\right) X_8, \tag{60}$$

where the prime indicates the second factor of $SO(16)$ and X_8 is explicitly given by

其中撇号表示 $SO(16)$ 的第二个因子，X_8 具体由下式给出

$$X_8 = \frac{1}{2}\left(\mathrm{ch}_4 + \mathrm{ch}_4'\right) - \frac{1}{48}\left(\mathrm{ch}_2^2 + \mathrm{ch}_2'^2 - \mathrm{ch}_2 \mathrm{ch}_2'\right). \tag{61}$$

Since the SO (16) × SO (16) heterotic string contains a rank-two antisymmetric tensor in its gravity multiplet, we can repeat the analysis presented above for the type-I string and conclude that all anomalies cancel through the GS mechanism.

由于 SO (16) × SO (16) 杂化弦的引力多重态中包含一个二阶反对称张量，我们可以重复前文对 I 型弦的分析，得出所有反常都通过 GS 机制消除的结论。

Type-I string theory with space-filling D- and anti-D-branes. We consider next the GS cancellation mechanism for type-IIB strings in the presence of one orientifold O9-plane and a stack of m D9-branes and n anti-D9- branes [25]. The gauge group is $\mathrm{SO}(m) \times \mathrm{SO}(n)$ and the charged massless chiral fields are a positive chirality fermion in the adjoint of $\mathrm{SO}(m)$, a positive chirality fermion in the symmetric tensor representation of $\mathrm{SO}(n)$, and a negative chirality spinor in the bifundamental of $\mathrm{SO}(m) \times \mathrm{SO}(n)$. These are excitations of open strings with endpoints attached to two D9-branes, two anti-D9-branes, and one D9-brane and one anti-D9-brane, respectively. Adding to this the contributions of the positive helicity gravitino and the negative helicity dilatino, the 12-form anomaly polynomial is given by

布满全空间的 D 膜和反 D 膜的 I 型弦论。我们接下来讨论存在一个 O9 定向宇称平面以及一堆 m 张 D9 膜和 n 张反 D9 膜时 [25]，IIB 弦的 GS 反常消除机制。规范群为 $\mathrm{SO}(m) \times \mathrm{SO}(n)$ ，带荷无质量手征场分别是: $\mathrm{SO}(m)$ 伴随表示中的正手性费米子、$\mathrm{SO}(n)$ 对称张量表示中的正手性费米子，以及 $\mathrm{SO}(m) \times \mathrm{SO}(n)$ 双基础表示中的负手性旋子。这些分别是端点连接在两张 D9 膜、两张反 D9 膜、一张 D9 膜一张反 D9 膜的开弦激发。加上正螺旋度引力微子和负螺旋度伸缩微子的贡献，12 形式反常多项式为

$$I_{12} = -\frac{1}{2} I_{\frac{1}{2}}[\mathcal{R}] + \frac{1}{2} I_{\frac{3}{2}}[\mathcal{R}] + \frac{1}{2} I_{\frac{1}{2}}[\mathcal{F}, \mathcal{R}]_{(\mathrm{adj}, \mathbf{1})}$$

$$+\frac{1}{2} I_{\frac{1}{2}}[\mathcal{F}, \mathcal{R}]_{(\mathbf{1}, \mathrm{sym})} - \frac{1}{2} I_{\frac{1}{2}}[\mathcal{F}, \mathcal{R}]_{(\mathrm{f}, \mathrm{f})}, \tag{62}$$

where the explicit expression for each term can be found in Eqs. (32), (33), and (39). We just look at the two terms spelling trouble for the implementation of the GS mechanism, which are the ones containing p_3 and ch_6 . The first one is proportional to

其中各项的具体表达式可见式 (32)、(32) 和 (39)。我们仅分析 GS 机制应用中存在问题的两项，即包含 p_3 和 ch_6 的两项。第一项正比于

$$496 - \frac{1}{2} m(m-1) - \frac{1}{2} n(n+1) - mn = 496 - \frac{1}{2}(m-n)(m-n-1),$$

(63)

which cancels for $m - n = 32$. As to the second offending term, we search for identities relating the traces of $\mathcal{F}^6$ in the various representations to those in the fundamental. The relevant ones are

它会对 $m - n = 32$ 抵消。至于第二个反常项，我们寻找将不同表示中 $\mathcal{F}^6$ 的迹与基础表示中的迹联系起来的恒等式，相关恒等式为

$$\mathrm{tr}_{(\mathrm{adj}, 1)} \mathcal{F}^6 = (n-32)\, \mathrm{tr}_{\mathrm{f}} \mathcal{F}^6 + 15 \left(\mathrm{tr}_{\mathrm{f}} \mathcal{F}^2\right)\left(\mathrm{tr}_{\mathrm{f}} \mathcal{F}^4\right),$$

$$\mathrm{tr}_{(\mathbf{1},\mathrm{sym})}\mathcal{F}^6 = (m+32)\,\mathrm{tr}_\mathrm{f}\mathcal{F}'^6 + 15\left(\mathrm{tr}_\mathrm{f}\mathcal{F}'^2\right)\left(\mathrm{tr}_\mathrm{f}\mathcal{F}'^4\right), \tag{64}$$

$$\mathrm{tr}_{(\mathrm{f},\mathrm{f})}\mathcal{F}^6 = n\mathrm{tr}_\mathrm{f}\mathcal{F}^6 + m\mathrm{tr}_\mathrm{f}\mathcal{F}'^6,$$

where the primes indicate the field strength associated to the $\mathrm{SO}(n)$ factor. Adding these three contributions with their respective signs, we find a cancellation of the irreducible terms proportional to $\mathrm{tr}\,\mathcal{F}_1^6$ and $\mathrm{tr}\,\mathcal{F}_2^6$. This ensures the factorization of the anomaly polynomial

其中素数表示与 $\mathrm{SO}(n)$ 因子相关联的场强。将这三项按各自的符号相加后，我们发现正比于 $\mathrm{tr}\,\mathcal{F}_1^6$ 和 $\mathrm{tr}\,\mathcal{F}_2^6$ 的不可约项被抵消了。这保证了反常多项式的因式分解

$$I_{12} = \left(p_1 - \frac{1}{30}\mathrm{ch}_2 - \frac{1}{30}\mathrm{ch}_2'\right)Y_8 \tag{65}$$

where

其中

$$Y_8 = -\frac{1}{48}\left[\mathrm{ch}_4 - \mathrm{ch}_4' - \frac{1}{60}p_1\left(\mathrm{ch}_2 - \mathrm{ch}_2'\right) + \frac{3}{8}p_1^2 - \frac{1}{2}p_2\right], \tag{66}$$

and the prime again indicates the Chern characters associated with $\mathrm{SO}(n)$. All gauge, gravitational, and mixed anomalies thus cancel for $m-n=32$ through the GS mechanism. As in the case of the type-I string, we also need to modify the antisymmetric tensor field strength

素数再次表示与 $\mathrm{SO}(n)$ 相关的陈特征。因此所有规范反常、引力反常和混合反常都通过 GS 机制对 $m-n=32$ 完全抵消。和 I 型弦的情况一样，我们还需要修正反对称张量场强

$$H = dC_2 - \frac{1}{30}\omega_3^0 - \frac{1}{30}\omega_3'^0 + \Omega_3^0 \tag{67}$$

and add the corresponding GS counterterm to the action

并在作用量中添加对应的 GS 抵消项

$$\Gamma_\mathrm{ct} = c\int_{\mathcal{M}_{10}} C_2 Y_8 + \frac{1+\alpha}{2}c\int_{\mathcal{M}_{10}}\left(\frac{1}{30}\omega_3^0 + \frac{1}{30}\omega_3'^0 - \Omega_3^0\right)Y_7^0, \tag{68}$$

with $Y_8 = dY_7^0$. ▯

其中 $Y_8 = dY_7^0$ 。▯

A four-dimensional version of the GS mechanism. As a final example, we study the implementation of the GS mechanism in a purely field-theoretical setup. Let us consider the theory of a positive chirality Weyl fermion coupled to a U(1) gauge field propagating, for simplicity, on flat spacetime. The corresponding anomaly polynomial is

四维版本的 GS 机制。作为最后一个例子，我们研究 GS 机制在纯场论框架下的实现。为简单起见，我们考虑正手征外尔费米子耦合到传播在平坦时空上的 U(1) 规范场的理论，对应的反常多项式为

$$I_6 = \mathrm{ch}_3, \tag{69}$$

and the theory has an anomaly given by

该理论的反常由下式给出

$$\delta_\eta S_{\text{eff}} = 2\pi i \int \omega_4^1 = -\int \eta \mathrm{ch}_2, \tag{70}$$

where we have used that ω_4^1 is proportional to $\mathcal{F}^2$ and therefore to the second Chern character.

这里我们用到了 ω_4^1 正比于 $\mathcal{F}^2$，因此正比于第二陈特征。

For a U(1) gauge group, the anomaly polynomial (69) trivially factors

对于 U(1) 规范群，反常多项式 (69) 可以平凡分解

$$\mathrm{ch}_3 = \frac{1}{3}\mathrm{ch}_1\mathrm{ch}_2 \tag{71}$$

and the GS mechanism is implemented by an axion field θ with the gauge transformation

GS 机制通过轴子场 θ 实现，该轴子场的规范变换为

$$\delta_\eta \theta = m_{\mathrm{GS}}\eta \tag{72}$$

and a gauge-invariant kinetic term

且具有规范不变的动能项

$$S \supset \frac{1}{2}\int (d\theta - m_{\mathrm{GS}}\mathcal{A}) \wedge \star (d\theta - m_{\mathrm{GS}}\mathcal{A}). \tag{73}$$

Here, m_{GS} is a constant with dimensions of mass. The anomaly is then canceled by the addition of the four-dimensional GS counterterm

此处 m_{GS} 是一个具有质量量纲的常数。通过添加四维 GS 抵消项即可抵消反常

$$S_{\mathrm{ct}} = -\frac{1}{m_{\mathrm{GS}}}\int \theta \mathrm{ch}_2 = \frac{1}{8\pi^2 m_{\mathrm{GS}}}\int \theta \mathcal{F}^2. \tag{74}$$

Indeed, $\delta_\eta S_{\text{eff}} + \delta_\eta S_{\text{ct}} = 0$ and gauge invariance is preserved quantum-mechanically.

事实上，$\delta_\eta S_{\text{eff}} + \delta_\eta S_{\text{ct}} = 0$，且规范不变性在量子力学层面得以保持。

This cancellation can also be understood in diagrammatic terms by noticing that (73) introduces a kinetic mixing between the axion and the gauge field of the form $\mathcal{A}^\mu \partial_\mu \theta$. On the other hand, the counterterm (74) induces a trivalent vertex with one axion and two gauge fields. Thus, the triangle diagram producing the anomaly (70) is canceled by a tree-level diagram where a gauge field transmutes into an axion that then decays into two other gauge fields. This is the four-dimensional analog of the diagram shown on the right panel of Fig. 1. In fact, the axion is just a two-form field in disguise, since both fields are related by Hodge duality in four dimensions, $d\theta = \star dB$ [26].

这种抵消也可以从费曼图的角度理解: 注意到 (73) 式已经在轴子和规范场之间引入了形式为 $\mathcal{A}^\mu \partial_\mu \theta$ 的动能混合；另一方面，抵消项 (74) 会诱导出一个包含一个轴子和两个规范场的三价顶点。因此，产生反常 (70) 的三角图会被一个树图抵消: 该树图中一个规范场转化为轴子，轴子再衰变为另外两个规范场。这就是图 1 右侧图的四维类比。实际上轴子只是伪装成二形式场，因为在四维中二者通过霍奇对偶联系，即 $d\theta = \star dB$ [26]。

Anomaly Inflow

反常流入

Having focused so far on anomaly cancellation, our discussion has avoided the evaluation of currents. In the gauge case, the (consistent) current is obtained by taking variations of the nonlocal quantum effective action Γ_{eff} with respect to the gauge one-form $\mathcal{A}$. The result splits into two pieces, one given by an integral over the "bulk" $\mathcal{M}_{2n-1}$ space and a second defined on its $(2n-2)$ -dimensional boundary12

迄今为止我们的讨论都聚焦于反常消除，避开了流的计算问题。在规范情形下，(相容) 反常可通过对非局域量子有效作用量 Γ_{eff} 关于规范一形式 $\mathcal{A}$ 变分得到。结果可拆分为两部分: 一部分是对“体” $\mathcal{M}_{2n-1}$ 空间的积分，另一部分定义在它的 $(2n-2)$ 维边界上 [12]

$$\delta\Gamma_{\text{eff}} = \int_{\mathcal{M}_{2n-1}} \mathrm{Tr}\,(J_{\text{bulk}}\,\delta\mathcal{A}) + \int_{\partial\mathcal{M}_{2n-1}} \mathrm{Tr}\,(X\delta\mathcal{A})\,. \tag{75}$$

The quantities inside the traces in both integrals are explicitly given by

两个积分中迹内的量可以显式写为

$$J_{\text{bulk}} = \frac{i^n}{(n-1)!(2\pi)^{n-1}}\mathcal{F}^{n-1}, \tag{76}$$

$$X = \frac{i^n}{(n-1)!(2\pi)^{n-1}}\int_0^1 tdt\left(\mathcal{F}_t^{n-2}\mathcal{A} + \mathcal{F}_t^{n-3}\mathcal{A}\mathcal{F}_t + \ldots + \mathcal{A}\mathcal{F}_t^{n-2}\right),$$

with $\mathcal{F}_t \equiv t\mathcal{F} + t\,(t-1)\,\mathcal{A}^2$. We can particularize (75) to the case of infinitesimal gauge transformations, $\delta_\chi \mathcal{A} = d\chi + [\mathcal{A}, \chi] \equiv D\chi$, where D denotes the gauge covariant derivative. Applying the Stokes theorem, we find the gauge variation of the effective action

其中满足 $\mathcal{F}_t \equiv t\mathcal{F} + t\,(t-1)\,\mathcal{A}^2$ 。我们可以将 (75) 特殊化到无穷小规范变换 $\delta_\chi \mathcal{A} = d\chi + [\mathcal{A}, \chi] \equiv D\chi$ 的情形，式中 D 表示规范协变导数。应用斯托克斯定理后，我们得到有效作用量的规范变分

$$\delta_\chi \Gamma_{\text{eff}} = -\int_{\mathcal{M}_{2n-1}} \operatorname{Tr}(\chi D J_{\text{bulk}}) + \int_{\partial\mathcal{M}_{2n-1}} \operatorname{Tr}[\chi(J_{\text{bulk}} - DX)]. \tag{77}$$

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[12] A direct way of getting this result is by applying the generalized transgression formula of [27] to the gauge Chern-Simons form ω^0_{2n-1}, using the one-parameter family of connections defined by $\mathcal{A}_t = \mathcal{A} + t\delta A$ (see [28] for details).

[12] 得到该结果的一种直接方法是将文献 [27] 的广义推移公式应用到规范陈-西蒙斯形式 ω^0_{2n-1}，使用由 $\mathcal{A}_t = \mathcal{A} + t\delta A$ 定义的单参数联络族 (细节见 [28])。

As we know, the left-hand side of this equation gives the consistent anomaly [cf. (1)], which we express in terms of the consistent current J_{cons} as[13]

如我们所知，该等式的左端给出了相容反常 [参见 (1)]，我们用相容流 J_{cons} as[13] 将其表示为

$$\delta_\chi \Gamma_{\text{eff}} \equiv -\int_{\partial\mathcal{M}_{2n-1}} \operatorname{Tr}(\chi D J_{\text{cons}}). \tag{78}$$

As to the right-hand side of (77), the first thing to notice is that the Bianchi identity $D\mathcal{F} = 0$ implies $DJ_{\text{bulk}} = 0$ so the first, nonlocal term vanishes. In addition to this, the quantity X defined in (76) is minus the Bardeen-Zumino term [4] relating the consistent and covariant currents, $J_{\text{cov}} = J_{\text{cons}} - X$ (see our discussion in page 2246). We thus obtain a very suggestive expression

对于 (77) 的右端,首先注意到比安基恒等式 $D\mathcal{F} = 0$ 蕴含 $DJ_{\text{bulk}} = 0$,因此第一项非局域项等于零。除此之外, (76) 中定义的量 X 正是联系相容流与协变流的巴丁-祖米诺项的负号 [4], 即 $J_{\text{cov}} = J_{\text{cons}} - X$ (参见我们第 2246 页的讨论)。由此我们得到一个十分有启发的表达式

$$DJ_{\text{cov}} = -J_{\text{bulk}}|_{\partial\mathcal{M}_{2n-1}}, \tag{79}$$

stating that the covariant anomaly equals minus the bulk current evaluated at the boundary.

它表明协变反常等于体流在边界上求值的负值。

To understand the implications of this relation, let us take a step back and review what we have done. By expressing it as an integral over a higher-dimensional space, the nonlocal effective action Γ_{eff} can be interpreted as describing the local quantum effective dynamics of gauge fields interacting with Dirac fermions in the odd-dimensional bulk Euclidean spacetime $\mathcal{M}_{2n-1}$. This theory is free of gauge anomalies, as it is manifest in the fact that the quantum current J_{bulk} is conserved, $DJ_{\text{bulk}} = 0$. The situation is quite different on the even-dimensional boundary theory, which contains chiral fermions and where DJ_{cov} (or DJ_{cons} for that matter) has a nonvanishing value, signaling the existence of a gauge anomaly. Equation (79) provides the clue to understand physically what is going on: the gauge anomaly results from the inflow of gauge charge from the bulk onto the boundary, as shown by the fact that the value of the bulk current there precisely cancels the rate of nonconservation of the gauge charge.

为了理解这个关系的含义，我们先退一步回顾我们做过的工作。通过将非局域有效作用量 Γ_{eff} 表示为高维空间上的积分，它可以被解释为描述与奇维体欧几里得时空 $\mathcal{M}_{2n-1}$ 中狄拉克费米子相互作用的规范场的局域量子有效动力学。该理论没有规范反常，这一点清楚体现在量子流 J_{bulk} 守恒这一事实上，即 $DJ_{\text{bulk}} = 0$。偶维边界理论的情况则完全不同，它包含手征费米子，且 DJ_{cov} (或者说 DJ_{cons}) 取值非零，预示着规范反常的存在。方程 (79) 为理解物理过程提供了线索: 规范反常来自规范荷从体到边界的流入，体流在边界的取值恰好抵消了规范荷不守恒的速率，这一事实印证了上述结论。

This general argument illustrates the basic features of the mechanism of anomaly inflow first pointed out in [29] (see also [3] for a review). Anomalous theories can be embedded in higher-dimensional, anomaly-free theories so that violations in the conservation of charge are accounted for by the inflow of charge from the higher-dimensional bulk. Or the other way around, nonanomalous theories in the presence of topological defects may have anomalies supported on their worldvolumes, which are nevertheless canceled by the charge flowing from outside the defect.

这个一般性论证阐明了最初在 [29] 中提出的反常流入机制的基本特征 (综述另见 [3])。反常理论可以嵌入高维无反常理论，此时电荷守恒的破坏可由来自高维体的电荷流入解释。反过来，无反常理论在存在拓扑缺陷时，缺陷世界体积上会出现反常，这类反常也可以通过从缺陷外部流入的电荷消除。

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[13] To simplify the discussion, we work for the time being with the Hodge duals of all one-form currents.

[13] 为简化讨论，我们暂时对所有一形式流都取霍奇对偶进行计算。

A simple example of anomaly inflow

反常流入的简单例子

Let us see the workings of anomaly inflow in a particular example, that of a massless Dirac fermion in 3+1 dimensions in the presence of an axion string defect [3, 29, 30] . The relevant terms in the action are

让我们通过一个具体实例了解反常流入的工作机制: 存在轴子弦缺陷 [3, 29, 30] 时，3+1 维中的无质量狄拉克费米子。作用量中的相关项为

$$S = \int d^4x \left[i\overline{\psi}\gamma^\mu \partial_\mu \psi + \overline{\psi}\left(\varphi_1 + i\gamma_5\varphi_2\right)\psi \right]. \tag{80}$$

Here, $\varphi \equiv \varphi_1 + i\varphi_2$ is a complex scalar field in the vacuum configuration

此处，$\varphi \equiv \varphi_1 + i\varphi_2$ 是真空构型中的复标量场

$$\varphi(x) = f(\rho)\, e^{i\theta(x)}, \tag{81}$$

where $\rho^2 = \left(x^1\right)^2 + \left(x^2\right)^2$ and $f(\rho)$ satisfies $f(\rho \to 0) = 0$ and $f(\rho \to \infty) = v$. This describes a stringlike vortex localized along the $x^3 \equiv z$ direction.

其中 $\rho^2 = \left(x^1\right)^2 + \left(x^2\right)^2$ 和 $f(\rho)$ 满足 $f(\rho \to 0) = 0$ 和 $f(\rho \to \infty) = v$ 。这描述了一个局域在 $x^3 \equiv z$ 方向的弦状涡旋。

The massless Dirac fermion contains two opposite chiralities and can be coupled to an electromagnetic U(1) without gauge anomalies. The question however is whether the string defect supports chiral fermion modes that might induce anomalies on its worldvolume. To clarify the issue, we split the Dirac fermion into its two chiralities $\gamma_5\psi_\pm = \pm\psi_\pm$ and separate the string worldvolume directions $x^a \equiv \left(x^0, x^3\right)$ from the transverse coordinates $\left(x^1, x^2\right) = (\rho\cos\phi, \rho\sin\phi)$. We also factorize the four-dimensional chirality matrix $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ as $\gamma_5 = \overline{\gamma}\gamma_T$, where $\overline{\gamma} = -\gamma^0\gamma^3$ is the two-dimensional chirality matrix and $\gamma_T \equiv -i\gamma^1\gamma^2$. Doing all this, the Dirac equation is recast as the pair of equations

无质量狄拉克费米子包含两种相反手征，可以在没有规范反常的情况下与电磁 U(1) 耦合。但问题在于，弦缺陷本身是否承载可能在其世界体积上诱导反常的手征费米子模式。为厘清这一问题，我们将狄拉克费米子拆分为两种手征 $\gamma_5\psi_\pm = \pm\psi_\pm$ ，并将弦世界体积方向 $x^a \equiv \left(x^0, x^3\right)$ 与横坐标 $\left(x^1, x^2\right) = (\rho\cos\phi, \rho\sin\phi)$ 分离。我们还将四维手征矩阵 $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ 分解为 $\gamma_5 = \overline{\gamma}\gamma_T$ ，其中 $\overline{\gamma} = -\gamma^0\gamma^3$ 是二维手征矩阵，且 $\gamma_T \equiv -i\gamma^1\gamma^2$ 。完成这些拆分后，狄拉克方程可改写为如下方程组

$$i\gamma^a\partial_a\psi_\pm + i\gamma^2\left(\cos\phi + i\overline{\gamma}_T\sin\phi\right)\partial_\rho\psi_\pm = f(\rho)\, e^{\mp i\phi}\psi_\mp, \tag{82}$$

admitting the solution

存在解

$$\psi_- = \eta\left(x^a\right)\exp\left[-\int_0^\rho d\rho' f\left(\rho'\right)\right],$$

$$\psi_+ = -i\gamma^2\psi_- \tag{83}$$

Here η is a negative chirality spinor satisfying the two-dimensional Dirac equation on the string worldvolume

此处 η 是满足弦世界体积上二维狄拉克方程的负手征旋量

$$i\gamma^a\partial_a\eta = 0,\ \overline{\gamma}\eta = -\eta. \tag{84}$$

We have shown the existence of a zero mode in the bulk Dirac equation corresponding to a massless spinor along the string defect whose chirality is correlated with its direction of propagation, $(\partial_0 - \partial_3)\,\eta = 0$. As a consequence, when coupling the theory to an external electromagnetic field $\mathcal{A}_\mu$, the chiral zero mode triggers a gauge anomaly on the string worldvolume

我们已经证明，体狄拉克方程中存在零模，对应弦缺陷上的无质量旋量，其手征与传播方向相关联，即 $(\partial_0 - \partial_3)\eta = 0$ 。因此，当该理论耦合到外电磁场 $\mathcal{A}_\mu$ 时，手征零模会在弦世界体积上引发规范反常

$$\partial_a J^a = \frac{e}{4\pi}\varepsilon^{ab}\mathcal{F}_{ab} = \frac{e\mathcal{E}}{2\pi}, \tag{85}$$

with $\mathcal{E}$ the external electric field along x^3 . This expression gives the amount of charge nonconservation per unit time and unit length. Notice that (85) gives the covariant anomaly, which in two dimensions is one-half the value of the consistent anomaly (for an explicit calculation of the consistent anomaly in this case, see, for example, [1]). Let us recall that this happens on a string defect otherwise embedded in a theory that as a whole is free from gauge anomalies.

其中 $\mathcal{E}$ 是沿 x^3 的外电场。该式给出单位时间、单位长度的电荷不守恒量。注意 (85) 给出的是协变反常，在二维中其值是自洽反常的一半 (该情况下自洽反常的具体计算可参见例如文献 [1])。我们要记得，该反常出现在弦缺陷上，而嵌入该缺陷的整个整体理论本身是没有规范反常的。

To give a physical picture of what is going on, we study the bulk theory outside the string defect and compute the one-loop corrected gauge current in the presence of the soliton φ . In the adiabatic approximation where scalar field gradients are small, this is given by the Goldstone-Wilczek current [31]

为了给发生的过程建立物理图像，我们研究弦缺陷外的体理论，计算孤子 φ 存在时一圈修正的规范流。在标量场梯度很小的绝热近似下，该流由戈德斯通-维尔切克流给出 [31]

$$\langle J^\mu \rangle_{\text{bulk}} = -\frac{ie}{16\pi^2}\varepsilon^{\mu\upsilon\alpha\beta}\frac{\varphi^*\partial_\upsilon\varphi - \varphi\partial_\upsilon\varphi^*}{|\varphi|^2}\mathcal{F}_{\alpha\beta}. \tag{86}$$

Evaluating it on the vacuum solution (81) and in the region where $f(\rho)$ is well approximated by its asymptotic value, we get

将其代入真空解 (81)，并在 $f(\rho)$ 可很好近似为其渐近值的区域计算，我们得到

$$\langle J^\mu \rangle_{\text{bulk}} = \frac{e}{8\pi^2}\varepsilon^{\mu\upsilon\alpha\beta}\partial_\upsilon\theta\mathcal{F}_{\alpha\beta}. \tag{87}$$

To clarify the workings of anomaly inflow in this case, let us take a gauge configuration describing an electric field $\mathcal{E}$ along $x^3 = z$. Thus, $\mathcal{F}_{ab} = \mathcal{E}\varepsilon_{ab}$ with all remaining components of the field strength equal to zero. The bulk electric current can then be written as

为厘清本例中反常流入的工作机制,我们取描述沿 $x^3 = z$ 方向电场 $\mathcal{E}$ 的规范构型。因此有 $\mathcal{F}_{ab} = \mathcal{E}\varepsilon_{ab}$ ，场强其余分量均为零。此时体电流可写为

$$\langle \mathbf{J} \rangle_{\text{bulk}} = -\frac{e\mathcal{E}}{4\pi^2}(\nabla\theta) \times \mathbf{u}_z. \tag{88}$$

Moreover, for an axion string solution we have $\theta(x) = \phi$, so the previous result takes the simpler form

此外，对于轴子弦解，我们有 $\theta(x) = \phi$ ，因此上述结果可简化为

$$\langle \mathbf{J} \rangle_{\text{bulk}} = -\frac{e\varepsilon}{4\pi^2\rho}\mathbf{u}_\rho. \tag{89}$$

This shows the existence of radial charge transport from the bulk towards the defect. Computing the flux of the bulk current (89) through a cylindrical surface centered on the string, we find the incoming charge per unit length and unit time to be

这表明存在从体流向缺陷的径向电荷输运。计算体流 (89) 穿过以弦为中心的圆柱面的通量，我们得到单位长度单位时间内的入射电荷为

$$\frac{dQ}{dtdL} = \frac{e\mathcal{E}}{2\pi}, \tag{90}$$

which exactly reproduces the rate of violation of electric charge conservation on the string worldvolume, as given in Eq. (87). The gauge anomaly on the defect is thus canceled by the inflow of charge from the bulk.

这完全重现了弦世界面上电荷守恒的破坏速率，即式 (87) 给出的结果。因此缺陷上的规范反常被体流入的电荷抵消了。

Our presentation here has closely followed Ref. [29]. It is possible nevertheless to recast the analysis in a language similar to the one used at the beginning of this section. We start with the bulk effective action describing an axion field coupled to the electromagnetic field

本文的表述紧密参考了文献 [29]。不过我们也可以用本节开头所用的类似语言重新进行分析。我们从描述轴子场与电磁场耦合的体有效作用量出发

$$S_{\text{axion}} = -\frac{e^2}{8\pi^2}\int_V \theta\mathcal{F}^2 \tag{91}$$

and take variations with respect to the gauge field

然后对规范场做变分

$$\delta S_{\text{axion}} = \frac{e^2}{4\pi^2}\int_V d\theta\mathcal{F}\delta\mathcal{A} - \frac{e^2}{4\pi^2}\int_{\partial V}\theta\mathcal{F}\delta\mathcal{A}. \tag{92}$$

Here we assumed that all fields go to zero at infinity and take ∂V a cylindrical surface surrounding the axion string. The expression obtained exhibits the structure displayed in Eq. (75), so we identify

此处我们假设所有场在无穷远处趋于零，并取 ∂V 为包围轴子弦的圆柱面。所得表达式展现出式 (75) 给出的结构，因此我们得到

$$J_{\text{bulk}} = \frac{e}{4\pi^2}d\theta\mathcal{F}, \tag{93}$$

where we absorbed a power of e into $\delta\mathcal{A}$. This is equivalent to Eq. (87) and using (79) we retrieve the two-dimensional covariant anomaly (85) [32]. I

其中我们将一个 e 幂次吸收到了 $\delta\mathcal{A}$ 中。这等价于式 (87)，利用 (79) 我们可以得到二维协变反常 (85)[32]。

Anomalous D-Brane Couplings

反常 D 膜耦合

Due to its universal nature, anomaly inflow is at work in a wide range of physical situations, ranging from condensed matter and fluid dynamics to lattice field theory and string theory. As to the last field, Polchinski's discovery of D-branes [33] as the sources of R-R charge not only solved a long-standing riddle. It also put the focus on the plethora of extended solitonic defects present in string theory, many of which had been already studied in the realm of supergravity (see [34] for a contemporary review on the issue). It is only natural that anomaly inflow found immediate application in this context.

由于反常流入具有普适性，它广泛作用于多种物理场景，从凝聚态物理、流体动力学到格点场论和弦理论都有涉及。在弦理论中，波利钦斯基发现 D 膜是 R-R 荷的源，这不仅解决了一个悬而未决的长期谜题，还让人们关注到弦理论中大量存在的延展孤子缺陷——其中许多早在超引力领域就已经被研究了 (相关当代综述见 [34])。反常流入自然很快就被应用到了这个场景中。

In a consistent string theory, any anomaly supported on the worldvolume of a defect (D-branes, orientifolds, NS-branes,...) has to be canceled by charge inflow from the bulk. The condition for this to happen determines the gauge and gravitational couplings of the extended object [35,36] (see also [13,37] for reviews). Let us see how this works in a case that looks very much like a generalization of the axion string discussed in page 2268. We consider a $(p+1)$ -form field C_{p+1} coupled to a p -brane embedded in a D -dimensional flat space $\mathcal{M}_D$. The relevant terms in the bulk low-energy action are

在自洽的弦理论中，所有支撑在缺陷世界体积上的反常 (包括 D 膜、orientifold、NS 膜等) 都必须通过体空间的荷流入抵消。该抵消条件决定了延展物体的规范耦合与引力耦合 [35,36]，综述参见 [13,37]。我们来看一个具体例子，它很像是第 2268 页讨论过的轴子弦的推广。我们考虑一个 $(p+1)$ 形式场 C_{p+1} ，与嵌入在 D 维平直空间 $\mathcal{M}_D$ 中的 p 膜耦合。体空间低能作用量中的相关项为

$$S \supset -\frac{1}{4}\int_{\mathcal{M}_D} H_{p+2} \wedge \star H_{p+2} + \frac{\mu_p}{2}\int_{\mathcal{B}_p} C_{p+1}, \tag{94}$$

where $H_{p+2} = dC_{p+1}$ is the gauge-invariant field strength and $\mathcal{B}_p$ the $(p+1)$-dimensional brane worldvolume. The associated equations of motion show that C_{p+1} is sourced by the p -brane

其中 $H_{p+2} = dC_{p+1}$ 是规范不变场强，$\mathcal{B}_p$ 是 (p+ 膜的 ($ (p+ -1$) 维世界体积。对应的运动方程表明，$C_{p+1}$ 由 p 膜作为源产生

$$d\star H_{p+2} = \mu_p \delta_{D-p-1}\left(\mathcal{B}_{p+1} \hookrightarrow \mathcal{M}_D\right), \tag{95}$$

where we introduced a $(D-p-1)$ -form delta function satisfying 14

这里我们引入了一个满足条件 (14) 的 $(D-p-1)$ 形式德尔塔函数

$$\int_{\mathcal{M}_D} \delta_{D-p-1}\left(\mathcal{B}_p \hookrightarrow \mathcal{M}_D\right) \alpha_{p+1} = \int_{\mathcal{B}_p} \alpha_{p+1}. \tag{96}$$

for any $(p+1)$ -form α_{p+1} defined on $\mathcal{B}_{p+1}$. A look at Eq. (95) indicates that the Hodge dual of this object can be interpreted as a worldvolume-supported $(p+1)$ - form current J_{p+1}

对任意定义在 $\mathcal{B}_{p+1}$ 上的 $(p+1)$ 形式 α_{p+1} 成立。观察式 (95) 可知，该对象的霍奇对偶可以解释为支撑在世界体积上的 $(p+1)$ 形式流 J_{p+1}

$$\star J_{p+1} = \delta_{D-p-1}\left(\mathcal{B}_p \hookrightarrow \mathcal{M}_D\right), \tag{97}$$

so the equations of motion for C_{p+1} are recast as $d \star H_{p+2} = \mu_p \star J_{p+1}$.

因此 C_{p+1} 的运动方程可以改写为 $d \star H_{p+2} = \mu_p \star J_{p+1}$ 。

Let us take p to be odd and assume that the p -brane worldvolume supports a chiral fermion zero mode leading to gauge anomalies. From Eq. (10) we know that the anomalous variation of the brane action is given by

设 p 为奇数，并假设 p 膜世界体积上存在手征费米零模，从而产生规范反常。由式 (10) 我们知道，膜作用量的反常变分由下式给出

$$\delta S_{\text{brane}} = 2\pi i \int_{\mathcal{B}_p} I^1_{p+1}, \tag{98}$$

where $dI^1_{p+1} = \delta I^0_{p+2}$, with I^0_{p+2} the Chern-Simons descended from the brane anomaly polynomial I_{p+3} . If the bulk theory on $\mathcal{M}_D$ is anomaly-free, its action should contain the coupling

其中 $dI^1_{p+1} = \delta I^0_{p+2}$ ，I^0_{p+2} 是从膜反常多项式 I_{p+3} 降维得到的陈-西蒙斯项。若 $\mathcal{M}_D$ 上的体理论无反常，其作用量应当包含如下耦合项

$$\Delta S_{\text{bulk}} = \frac{2\pi i}{\mu_p} \int_{M_D} \star H_{p+2} \wedge I^0_{p+2}, \tag{99}$$

[14] These delta functions are known as de Rham currents.

[14] 这些德尔塔函数被称为德拉姆流。

whose gauge variation cancels the anomaly on the brane (98)

其规范变分抵消了膜上的反常 (98)

$$\delta\Delta S_{\text{bulk}} = -\frac{2\pi i}{\mu_p}\int_{\mathcal{M}_D} d\star H_{p+2}\wedge I^1_{p+1} = -2\pi i\int_{\mathcal{B}_{p+1}} I^1_p. \tag{100}$$

In the second identity we have used the equations of motion (95). Just by requiring that anomaly inflow restores the consistency of theory we determined the coupling in Eq. (99). The other way around, a computation of the term (99) from the bulk theory determines μ_p, measuring the coupling of the p-brane to C_{p+1}.

在第二个恒等式中我们用到了运动方程 (95)。仅通过要求反常流入恢复理论的自洽性，我们就确定了式 (99) 中的耦合项。反之，从体理论出发计算项 (99)，就可以确定出测量 p 膜与 C_{p+1} 耦合的 μ_p。

We want to apply the philosophy of this calculation to the case of Dp-branes. We might be tempted to consider the vanilla case of a single flat Dp-brane in flat space (or a parallel stack of them). In type-IIB theory p takes odd values so the $(p+1)$-dimensional D-brane worldvolume is even-dimensional and may contain chiral fermions. Yet, there is the same number of positive and negative chirality fermions on the brane, all in the adjoint representation of the gauge group. This means that the theory is anomaly-free. Another way to understand the absence of gauge brane anomalies in this setup is by taking into account that the bulk type-IIB theory does not contain gauge fields. Were the theory on the brane anomalous, its anomaly would have no chance of being canceled by accretion/depletion of gauge charge from the ten-dimensional bulk.

我们希望将该计算思路应用到 Dp-膜的情况中。我们或许会想考虑平坦空间中单张平坦 Dp-膜 (或平行堆叠的 Dp-膜) 的普通情形。在 IIB 型理论中，p 取奇数，因此 $(p+1)$ 维 D 膜世界体是偶数维的，可能包含手征费米子。但膜上正负手征费米子数量相等，且都属于规范群的伴随表示。这说明该理论无反常。另一种理解该设置中不存在规范膜反常的方式是:IIB 型体理论本身不包含规范场。若膜上理论存在反常，该反常根本无法通过十维体中规范荷的增减来抵消。

We need to consider less simple configurations including also the effects of gravitational anomalies. One interesting scenario is that of intersecting branes whose combined worldvolumes span the whole ten-dimensional spacetime while producing a chiral theory at an even-dimensional intersection [35]. Anomalies there are compensated by the inflow of charge from the "parent" D-branes' worldvolumes. The condition for this cancellation to happen determines the anomalous gauge and gravitational couplings of the Dp-brane.

我们需要考虑更复杂的构型，同时还要考虑引力反常的效应。一个有趣的情形是相交膜: 其整体世界体覆盖整个十维时空，且在偶数维交截处产生手征理论 [35]。该处的反常可通过“母”D 膜世界体的荷流入抵消，该抵消成立的条件决定了 Dp-膜的反常规范耦合与引力耦合。

Another way to generate a chiral theory on a Dp-brane, with p odd, is by playing with curvature [36]. Let us consider a stack of N coincident Dp-branes whose worldvolume wrap $\mathcal{B}_{p+1}$. In this D-brane background the ten-dimensional local Lorentz group $\mathrm{SO}(1,9)$ breaks down to $\mathrm{SO}(1,p)\times\mathrm{SO}(9-p)$, where the first factor is the local Lorentz group on the brane worldvolume and the second the R-symmetry of the maximal SYM theory describing the dynamics on the brane. This breaking of the structure group of the tangent bundle $T\mathcal{M}$ reflects its decomposition into a Whitney sum of the brane tangent and normal bundles

当 p 为奇数时,在 Dp-膜上构造手征理论的另一种方法是引入曲率 [36]。考虑一堆共位置的 Dp-膜,其世界体缠绕 $\mathcal{B}_{p+1}$,共 N 张。在该 D 膜背景下,十维局域洛伦兹群 SO (1, 9) 破缺为 SO (1, p)×SO (9 − p) ，其中第一个因子是膜世界体上的局域洛伦兹群，第二个因子是描述膜动力学的最大超对称杨-米尔斯理论的 R 对称性。切丛结构群 $T\mathcal{M}$ 的这种破缺，对应其分解为膜切丛与法丛的惠特尼和。

$$TM = T\mathcal{B}_p \oplus TN \tag{101}$$

At the same time, fermions in the 16 Majorana-Weyl representation of Spin(1,9) split into representations of Spin $(1, p) \times$ Spin $(9 - p)$ with correlated chiral-ities. As an example, for $p = 3$ we have the $(\mathbf{2}, \mathbf{4}) \oplus (\overline{\mathbf{2}}, \overline{\mathbf{4}})$ representation of Spin $(1, 3) \times$ Spin (6) , whereas for $p = 5$ the 16 of Spin $(1, 9)$ decomposes as $(\mathbf{4}_p, \mathbf{2}_p) \oplus (\mathbf{4}'_p, \mathbf{2}'_p)$. Incidentally, the previous examples display an interesting general property: for $p = 1 \bmod 4$ the two spinor chiralities transform in independent representations of Spin $(9 - p)$, whereas when $p = 3 \bmod 4$ they are complex conjugate of each other (see, for example, [38]).

同时，Spin(1,9) 的 16 维马约拉纳-外尔表示中的费米子会分解为 Spin $(1, p) \times$ Spin $(9 - p)$ 的表示，且手征性相互关联。例如，当 $p = 3$ 时，我们得到 Spin $(1, 3) \times$ Spin (6) 的 $(\mathbf{2}, \mathbf{4}) \oplus (\overline{\mathbf{2}}, \overline{\mathbf{4}})$ 表示；而当 $p = 5$ 时，Spin $(1, 9)$ 的 16 维表示分解为 $(\mathbf{4}_p, \mathbf{2}_p) \oplus (\mathbf{4}'_p, \mathbf{2}'_p)$ 。顺带一提，上述例子体现了一个有趣的普遍性质: 当 $p = 1 \bmod 4$ 时，两种旋子手征性按 Spin $(9 - p)$ 的独立表示变换，而当 $p = 3 \bmod 4$ 时，它们互为复共轭 (例如参见 [38])。

The group Spin $(9 - p)$ is a gauge invariance of the theory whose gauge potential is given by the normal space components of the spin connection, ω_N . In addition, chiral fermions on the brane also have U (N) Chan-Paton quantum numbers. Being massless excitations of open string with endpoints lying on a brane of the stack, they transform in its $\mathbf{N} \otimes \overline{\mathbf{N}}$ representation, which is real and therefore ”safe” from the point of view of anomalies (in other words, U (N) transformations are chirality-blind). In the mathematical lingo what we have is a vector bundle $(V \otimes S_N^+) \oplus (V \otimes S_N^-)$, where V is the Chan-Paton bundle and $S_N^{\pm}$ are the spin bundles over the normal space associated with each chirality.

群 Spin $(9 - p)$ 是该理论的规范不变性，其规范势由自旋联络的法空间分量 ω_N 给出。此外，膜上的手征费米子还具有 U (N) 陈-帕顿量子数。作为端点位于叠堆中某一张膜上的开弦的无质量激发，它们在该叠堆的 $\mathbf{N} \otimes \overline{\mathbf{N}}$ 表示下变换，该表示是实表示，因此从反常的角度来看是“安全”的 (换句话说，U (N) 变换不区分手征)。用数学术语来说，我们得到的是一个向量丛 $(V \otimes S_N^+) \oplus (V \otimes S_N^-$ ，其中 V 是陈-帕顿丛，$S_N^{\pm}$ 是法空间上对应每种手征的自旋丛。

The only source of anomalies on the brane worldvolume are chiral fermions, so the anomaly polynomial is given by [15] [cf. (11)]

膜世界体积上反常的唯一来源是手征费米子，因此反常多项式由 [15] 给出 [参见式 (11)]

$$\begin{aligned} I &= \frac{1}{2}\widehat{A}(\mathcal{B}_p)\left[\mathrm{ch}\left(V \otimes S_N^+\right) - \mathrm{ch}\left(V \otimes S_N^-\right)\right] \\ &= \frac{1}{2}\widehat{A}(\mathcal{B}_p)\,\mathrm{ch}(V)\left[\mathrm{ch}\left(S_N^+\right) - \mathrm{ch}\left(S_N^-\right)\right], \end{aligned} \tag{102}$$

where the factor of $\frac{1}{2}$ in front accounts for Majorana-Weyl fermions and we have implemented the factorization property of the Chern character for product vector bundles, $\mathrm{ch}(U \otimes W) = \mathrm{ch}(U)\,\mathrm{ch}(W)$. The Chern character of the Chan-Paton bundle can be further rewritten as

其中前置因子 $\frac{1}{2}$ 对应马约拉纳-外尔费米子，我们已经对乘积向量丛的陈特征应用了分解性质，即 $\mathrm{ch}(U \otimes W) = \mathrm{ch}(U)\,\mathrm{ch}(W)$ 。陈-帕顿丛的陈特征可以进一步改写为

$$\mathrm{ch}(V) \equiv \mathrm{ch}_{\mathbf{N}\otimes\overline{\mathbf{N}}}(\mathcal{F}) = \mathrm{ch}_{\mathbf{N}}(\mathcal{F})\,\mathrm{ch}_{\overline{\mathbf{N}}}(\mathcal{F}) = \mathrm{ch}(\mathcal{F})\,\mathrm{ch}(-\mathcal{F})\,. \tag{103}$$

In the last equality we used the Hermitian character of the $\mathrm{U}(N)$ generators and dropped the subscript with the understanding that all traces are computed in the fundamental representation of $\mathrm{U}(N)$. With this, Eq. (102) is recast as

在上一个等式中，我们利用了 $\mathrm{U}(N)$ 生成元的厄米性，并且省略了下标，默认所有迹都是在 $\mathrm{U}(N)$ 的基础表示下计算的。据此，式 (102) 可改写为

$$I = \frac{1}{2}\widehat{A}\left(T\mathcal{B}_p\right)\mathrm{ch}(\mathcal{F})\,\mathrm{ch}(-\mathcal{F})\left[\mathrm{ch}\left(S_N^+\right) - \mathrm{ch}\left(S_N^-\right)\right]. \tag{104}$$

Here it is manifest that the anomaly results from the asymmetry between S_N^+ and S_N^- . It remains to see how this is related to the curvature of the normal bundle, $\mathcal{R}_N = d\omega_N + \omega_N^2$.

此处可以明显看出，反常来源于 S_N^+ 和 S_N^- 之间的不对称性。接下来我们需要说明这和法丛的曲率 $\mathcal{R}_N = d\omega_N + \omega_N^2$. 有何关联

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[15] To be clear, there are some subtleties if $p = 3 \mod 4$ when fermions with opposite chiralities transform in complex conjugate representations of $\mathrm{Spin}(9-p)$. This notwithstanding, the anomaly polynomial is the same as in the case $p = 1 \mod 4$ [36].

[15] 需要明确的是，当 $p = 3 \mod 4$ 相反手征的费米子在 $\mathrm{Spin}(9-p)$ 的复共轭表示下变换时，会存在一些微妙之处。尽管如此，反常多项式仍和 $p = 1 \mod 4$ 的情形一致 [36]。

To achieve this, we need an object that somehow we managed to do without so far, the Euler class. For a manifold $\mathcal{M}$ of dimension $2k$, the Euler class $e(\mathcal{M})$ is defined as the $2k$ -form

为此，我们需要一个此前一直没有用到的对象: 欧拉类。对于维度为 $2k$ 的流形 $\mathcal{M}$ ，欧拉类 $e(\mathcal{M})$ 被定义为 $2k$ 次形式

$$e(\mathcal{M}) = x_1 \dots x_k, \tag{105}$$

where x_i are again the skew eigenvalues introduced in Eq. (12). The Euler class is thus the Pfaffian of $\frac{1}{2\pi}\mathcal{R}^a{}_b$ or, equivalently, the square root of the maximal Pontrjagin class. With this new polynomial at hand, we apply the general relation (see, for example, [39])

其中 x_i 仍是式 (12) 中引入的斜对称本征值。因此欧拉类是 $\frac{1}{2\pi}\mathcal{R}^a{}_b$ 的普法夫式，等价于最高庞特里亚金类的平方根。得到这个新多项式后，我们应用一般关系式 (例如参见 [39])

$$\mathrm{ch}\left(S^+_{\mathcal{M}}\right) - \mathrm{ch}\left(S^-_{\mathcal{M}}\right) = \prod_{i=1}^{k}\left(e^{x_i/2} - e^{-x_i/2}\right) = \frac{e(\mathcal{M})}{\widehat{A}(\mathcal{M})} \tag{106}$$

and rewrite the anomaly polynomial on the brane (104) as

并将膜上的反常多项式 (104) 改写为

$$I = \frac{1}{2}\,\mathrm{ch}(\mathcal{F})\,\mathrm{ch}(-\mathcal{F})\,\frac{\widehat{A}\left(T\mathcal{B}_p\right)}{\widehat{A}(TN)}e(TN)\,. \tag{107}$$

Using the explicit expression of the A -roof in (17), the quotient in the previous equation can be expanded in terms of the Pontrjagin classes for the tangent and normal bundle, $p_i(\mathcal{R}_T) \equiv p_i$ and $p_i(\mathcal{R}_N) \equiv p'_i$,

利用 (17) 中 A 帽的显式表达式，上一方程中的商可以展开为切丛和法丛的庞特里亚金类 $p_i(\mathcal{R}_T) \equiv p_i$ 和 $p_i(\mathcal{R}_N) \equiv p'_i$ 的组合，

$$\frac{\widehat{A}\left(T\mathcal{B}_p\right)}{\widehat{A}(TN)} = 1 - \frac{1}{24}\left(p_1 - p'_1\right)$$

$$+\frac{1}{5760}\left(7p_1^2 + 3p_1'^2 - 10p_1p'_1 - 4p_2 + 4p'_2\right) + \ldots\,. \tag{108}$$

Notice how the chiral asymmetry is controlled by the curvature of the normal bundle: setting $\mathcal{R}_N = 0$ implies $e(TN) = 0$ and the anomaly polynomial vanishes.

可以注意到手征不对称性由法丛的曲率调控: 令 $\mathcal{R}_N = 0$ 则可得 $e(TN) = 0$ ，此时反常多项式为零。

We have completed half our task. Brane anomalies are then computed from the polynomial (107) following the standard descend method: we extract its $p+3$ -form piece $I_{p+3} = dI^0_{p+2}$ and get the anomaly as $\delta I^0_{p+2} = dI^1_{p+1}$. In a consistent theory this has to be canceled by charge inflow from the bulk.

我们已经完成了一半任务。接下来可以按照标准降维方法从多项式 (107) 计算膜反常: 我们提取出它的 $p+3$ 形式项 $I_{p+3} = dI^0_{p+2}$ ，得到反常为 $\delta I^0_{p+2} = dI^1_{p+1}$ 。在自洽的理论中，该反常必须被体空间流入的电荷抵消。

This is the condition we are imposing now to determine the couplings of the Dp -brane to the various R-R fields of type-IIB string theory. Let us define the field

这就是我们现在用来确定 Dp 膜与 IIB 型弦论各类 R-R 场耦合的条件。我们来定义该场

$$C \equiv \sum_{k=0}^{5} C_{2k} \tag{109}$$

and assume that its dynamics is described by the action

并假设它的动力学由如下作用量描述

$$S_{\mathrm{RR}} = -\frac{1}{4}\int_{\mathcal{M}_{10}} H \wedge \star H - \frac{1}{2}\sum_k \mu_k \int_{\mathcal{B}_k} C Y_k, \tag{110}$$

where $H \equiv dC$ is the field strength. In the previous ansatz we have introduced coupling constants μ_k to the different D-branes, while Y_k are invariant polynomials built from the brane gauge field strength and the curvature. Their overall normalization is chosen so they can be written as $Y_k = N_k + dY_k^0$, with N_k the number of D-branes wrapping on the worldvolume $\mathcal{B}_k$. After an integration by parts, we have

其中 $H \equiv dC$ 是场强。在之前的假设中，我们已经引入了不同 D 膜对应的耦合常数 μ_k ，而 Y_k 是由膜规范场强和曲率构造的不变多项式。我们选取了整体归一化使得它们可以写为 $Y_k = N_k + dY_k^0$ ，其中 N_k 是包裹在世界体积 $\mathcal{B}_k$ 上的 D 膜数目。分部积分后，我们得到

$$S_{\mathrm{RR}} = -\frac{1}{4}\int_{\mathcal{M}_{10}} H \wedge \star H$$

$$-\frac{1}{2}\sum_k \mu_k \int_{\mathcal{M}_{10}} \delta_{9-k}\left(\mathcal{B}_k \hookrightarrow \mathcal{M}_{10}\right)\left(N_k C - H Y_k^0\right), \tag{111}$$

where to express the second term on the right-hand side as an integral over the ten-dimensional bulk spacetime, we have used the $(9-k)$ -form delta function introduced in (96).

其中为了将右手边第二项表示为十维体时空上的积分，我们使用了 (96) 中引入的 $(9-k)$ 形式 δ 函数。

The action (111) leads to the equations of motion

作用量 (111) 导出运动方程

$$d \star H = \sum_k \mu_k \delta_{9-k}\left(\mathcal{B}_k \hookrightarrow \mathcal{M}_{10}\right) Y_k. \tag{112}$$

We have to remember however that in type-IIB theory C_{p+1} is dual to C_{9-p} (C_4 is self-dual, $H_5 = \star H_5$). This means that Dp -branes are electric sources of C_{p+1} , whose magnetic sources are D$(9-p)$ -branes. With this in mind, the Bianchi identity derived from the action (111) reads

但我们必须记得，在 IIB 型理论中 C_{p+1} 对偶于 C_{9-p} (C_4 是自对偶的，$H_5 = \star H_5$)。这意味着 Dp 膜是 C_{p+1} 的电源，其磁源是 D$(9-p)$ 膜。牢记这点，从作用量 (111) 导出的比安基恒等式可写为

$$dH = -\sum_k \mu_k \delta_{9-k}\left(\mathcal{B}_k \hookrightarrow \mathcal{M}_{10}\right) \bar{Y}_k, \tag{113}$$

where $\bar{Y}_k$ is obtained from Y_k by taking the complex conjugate of the (fundamental) Chan-Paton bundle. The conclusion is that the presence of the brane leads to a modification of the original field strength $H = dC$, which now picks up terms depending on the brane gauge field strength and curvature

其中 $\bar{Y}_k$ 由对 (基本) 陈-帕顿丛取复共轭从 Y_k 得到。结论是，膜的存在会修改原场强 $H = dC$ ，修改后的场强会包含依赖于膜规范场强和曲率的项

$$H = dC - \sum_k \mu_k \delta_{9-k} (\mathcal{B}_k \hookrightarrow \mathcal{M}_{10}) \bar{Y}_k^0. \tag{114}$$

Originally, the field C was gauge invariant. Requiring however that (114) remains invariant leads to the transformation

原本，场 C 是规范不变的。但要求 (114) 保持不变会导出如下变换

$$\delta C = \sum_k \mu_k \delta_{9-k} (\mathcal{B}_k \hookrightarrow \mathcal{M}_{10}) \bar{Y}_k^1, \tag{115}$$

where Y_k^1 is defined by $\delta Y_k^0 = dY_k^1$ and similarly for $\bar{Y}_k^1$. An important point here is that the modification of the gauge transformation of the C field is restricted to the brane worldvolumes.

其中 Y_k^1 由 $\delta Y_k^0 = dY_k^1$ 定义，$\bar{Y}_k^1$ 同理。这里一个重要的点是，C 场规范变换的修正仅局限于膜世界体积上。

We can compute now the gauge variation of the R-R action (111). Implementing the gauge transformation (115) and the Bianchi identity (113), we find

我们现在可以计算 R-R 作用量 (111) 的规范变分。代入规范变换 (115) 和比安基恒等式 (113)，我们得到

$$\delta S_{\mathrm{RR}} = -\frac{1}{2} \sum_{j,k} \mu_j \mu_k \int_{\mathcal{M}_{10}} \delta_{9-j} (\mathcal{B}_j \hookrightarrow \mathcal{M}_{10}) \delta_{9-k} (\mathcal{B}_k \hookrightarrow \mathcal{M}_{10}) \left(Y_j \overline{Y_k} \right)^{(1)},$$

(116)

where we have written

其中我们已经记

$$\left(Y_j \overline{Y_k} \right)^{(1)} \equiv N_j \bar{Y}_k^1 + Y_j^1 \bar{Y}_k. \tag{117}$$

The product of the two delta forms can be reduced to a single one taking into account the property [36]

利用性质 [36]，两个 δ 形式的乘积可以约化为单个 δ 形式

$$\delta_{9-j} (\mathcal{B}_j \hookrightarrow \mathcal{M}_{10}) \delta_{9-k} (\mathcal{B}_k \hookrightarrow \mathcal{M}_{10})$$

$$= \delta_{9-j} (\mathcal{B}_j \cap \mathcal{B}_k \hookrightarrow \mathcal{M}_{10}) e (TN_j \cap TN_k). \tag{118}$$

With this, the anomalous variation of S_{RR} can be expressed as

由此，S_{RR} 的反常变分可以写为

$$\delta S_{\mathrm{RR}}=-\frac{1}{2}\sum_{j,k}\mu_j\mu_k\int_{\mathcal{M}_{10}}\delta_{9-j}\left(\mathcal{B}_j\cap\mathcal{B}_k\hookrightarrow\mathcal{M}_{10}\right)e\left(TN_j\cap TN_k\right)\left(Y_j\overline{Y_k}\right)^{(1)},$$

(119)

which, as we see, is supported on the D-brane intersections.

正如我们所见，它支撑在 D 膜的交线上。

The notation used in Eq. (117) was not unintentional. The superscript on the left-hand side indicates that this term is obtained by descend from the invariant polynomial $Y_j\overline{Y_k}$

式 (117) 中使用的记号并非无意。左手边的上标表明该项是由不变多项式 $Y_j\overline{Y_k}$ 降维得到的

$$Y_j\bar{Y}_k=N_jN_k+d\left(N_j\bar{Y}_k^0+N_kY_j^0+Y_j^0d\bar{Y}_k^0\right)\equiv N_jN_k+d\left(Y_j\bar{Y}_k\right)^{(0)},$$

$$\delta\left(Y_j\bar{Y}_k\right)^{(0)}=d\left(N_j\bar{Y}_k^1+Y_j^1\bar{Y}_k\right)\equiv d\left(Y_j\bar{Y}_k\right)^{(1)}. \tag{120}$$

This means that the gauge variation (119) can be derived from the brane anomaly polynomial

这说明规范变分 (119) 可以从膜反常多项式导出

$$I'_{\mathrm{bulk}}=-\frac{1}{4\pi}\sum_{j,k}\mu_j\mu_kY_j\overline{Y_k}e\left(TN_j\cap TN_k\right), \tag{121}$$

where the factor of 4π comes from the global normalization of the anomaly.

其中因子 4π 来自反常的整体归一化

After all these calculations, we can finally make contact with the D-brane anomaly polynomial in (107). Particularizing the bulk anomaly polynomial (121) to a single stack of Dp -branes

完成所有这些计算后，我们最终可以与 (107) 中的 D 膜反常多项式联系起来。将体反常多项式 (121) 特殊化为单堆 Dp 膜

$$I'_{\mathrm{bulk}}=-\frac{\mu^2}{4\pi}Y\bar{Y}e(TN), \tag{122}$$

the condition that the anomalous variation of the bulk R-R action cancels the anomaly on the brane determines the invariant polynomial Y to be

体 R-R 作用量的反常变分抵消膜上反常的条件将不变多项式 Y 确定为

$$Y=-\frac{\sqrt{4\pi}}{\mu}\,\mathrm{ch}\,(\mathcal{F})\sqrt{\frac{\widehat{A}\left(T\mathcal{B}_p\right)}{\widehat{A}(TN)}}. \tag{123}$$

This fixes the Dp-brane R-R couplings in the action (110)

这就确定了作用量 (110) 中的 Dp 膜 R-R 耦合

$$S_{\text{int}} = T_p \int_{\mathcal{B}_p} C\,\text{ch}\left(\ell_s^2 \mathcal{F}\right) \sqrt{\frac{\widehat{A}\left(\ell_s^2 \mathcal{R}_T\right)}{\widehat{A}\left(\ell_s^2 \mathcal{R}_N\right)}}, \tag{124}$$

where, for the sake of clarity, we have indicated the dependence of the A -roof genera on the curvatures of the tangent and normal bundles. We have also restored the powers of the string length and the Dp -brane tension is defined as

为清晰起见，我们在此标出了 A 亏根对切丛与法丛曲率的依赖关系。我们还还原了弦长的幂次，Dp 膜张力求定义为

$$T_p \equiv \frac{2\pi}{\ell_s^{p+1}} = 2\pi\left(4\pi^2\alpha'\right)^{-\frac{p+1}{2}}. \tag{125}$$

It should be pointed out that although our analysis has focused on the type-IIB theory, the action (124) is valid as well in type-IIA string theory and even p . As an example, let us work out the case of a D3-brane. Extracting the terms of rank four in the integrand, we have

需要指出的是，尽管我们的分析聚焦于 IIB 型理论，但作用量 (124) 在 IIA 型弦理论甚至 p 中同样成立。我们以 D3 膜为例展开推导。提取被积函数中秩为 4 的项，我们得到

$$S_{\text{int}} = T_3 \int_{\mathcal{B}_3} \left[N C_4 + \frac{i\ell_s^2}{2\pi} C_2 \text{tr}_{\text{f}} \mathcal{F} + \frac{\ell_s^4}{384\pi^2} C_0 \left(N\, \text{Tr}\, \mathcal{R}^2 - 1440\, \text{tr}\, \mathcal{F}^2 \right) \right].$$

(126)

where, to make things simpler, we have also set $\mathcal{R}_N = 0$.

为了简化推导，我们在这里同样设定 $\mathcal{R}_N = 0$ 。

A similar analysis can be carried out for orientifold planes [40,41], where anomalies are originated in self-dual tensor fields. The resulting coupling is expressed using the Hirzebruch polynomial introduced in Eq. (31)

类似的分析也可以适用于 orientifold 平面对 [40,41]，这类平面的反常起源于自对偶张量场。得到的耦合可以用式 (31) 引入的希策布鲁赫多项式表示

$$S_{\text{int}} = -2^{p-4} T_p \int_{\mathcal{O}_p} C \sqrt{\frac{L\left(\ell_s^2 \mathcal{R}_T/4\right)}{L\left(\ell_s^2 \mathcal{R}_N/4\right)}}, \tag{127}$$

where the square root is expanded in terms of the Pontrjagin classes for the tangent and normal bundle

其中平方根按切丛和法丛的庞特里亚金类展开

$$\sqrt{\frac{L\left(\ell_s^2\mathcal{R}_T/4\right)}{L\left(\ell_s^2\mathcal{R}_N/4\right)}} = 1 + \frac{\ell_s^4}{96}\left(p_1 - p_1'\right) \tag{128}$$

$$+\frac{\ell_s^8}{92160}\left(448p_2 - 448p_2' - 91p_1^2 + 19p_1'^2 - 10p_1p_1'\right) + \dots .$$

Notice that since there are no open strings attached to the orientifold plane, only gravitational anomalies may arise.

注意，由于没有开弦连接到 orientifold 平面，因此仅可能出现引力反常。

These anomalous couplings have an interesting connection with the GS mechanism [40]. In the modern language, type-I string theory is formulated as type-IIB theory in the presence of 32 D9-branes and one orientifold O9-plane. We can use expressions (124) and (127) to compute how the R-R two-form C_2 and its dual C_6 couple to these space-filling defects. Adding their corresponding contributions and keeping the piece of rank 10, we find the relevant terms in the action to be[16]

这些反常耦合与 GS 机制存在值得关注的关联 [40]。用现代语言表述，I 型弦理论表述为存在 32 个 D9 膜和一个 orientifold O9 平面背景下的 IIB 型理论。我们可以利用表达式 (124) 和 (127) 计算 R-R 二形式 C_2 及其对偶 C_6 如何与这些满空间缺陷耦合。将它们对应的贡献相加并保留秩为 10 的项，我们得到作用量中对应 be[16] 的相关项为

$$S_{\text{int}} \supset \frac{2\pi}{\ell_s^6}\int_{\mathcal{M}_{10}}\left[C_6\left(\text{ch}_{\text{f},2} - p_1\right)\right.$$

$$\left.+\ell_s^4 C_2\left(\text{ch}_{\text{f},4} - \frac{1}{48}p_1\text{ch}_{\text{f},2} + \frac{1}{64}p_1^2 - \frac{1}{48}p_2\right)\right], \tag{129}$$

where in the Chern characters we indicated explicitly that traces are computed in the fundamental of SO (32) . The coupling of C_6 determines the Bianchi identity for C_2 to be $dH_3 = \ell_s^2\left(\text{ch}_{\text{f},2} - p_1\right)$, imposing the modification of the field strength $H_3 = dC_2 + \ell_s^2\left(\omega_3^0 - \Omega_3^0\right)$. This reproduces Eq. (57) with $B = C_2$. Moreover, writing the Chern characters explicitly in terms of traces, we recover as well the GS counterterm (58) from the second term on the right-hand side of (129). This shows that the GS mechanism can be regarded as an instance of anomaly inflow, where the anomaly in the open string sector is cancelled by gauge and gravitational charge influx provided by the closed string R-R rank-two antisymmetric tensor field, which now has gauge charge due to its modified field strength.

其中我们在陈特征中明确标出，迹是在 SO (32) 的基础表示下计算的。C_6 的耦合给出了 C_2 的比安基恒等式为 $dH_3 = \ell_s^2\left(\text{ch}_{\text{f},2} - p_1\right)$ ，要求对场强 $H_3 = dC_2 + \ell_s^2\left(\omega_3^0 - \Omega_3^0\right)$ 做修正。这在 $B = C_2$ 下重现了式 (57)。此外，将陈特征显式写为迹的形式，我们也从 (129) 右侧第二项中得到了 GS 抵消项 (58)。这说明 GS 机制可以看作反常入流的一个实例: 开弦 sector 的反常由闭弦 R-R 秩二反对称张量场提供的规范与引力荷入流抵消，该场因修正后的场强而带有规范荷。

A Modern Take on Anomalies

反常的现代视角

Our overview of anomalies has been restricted to those jeopardizing the invariance of quantum theories under infinitesimal diffeomorphisms and gauge transformations, which for this very reason are visible in perturbation theory. But anomalies can also affect transformations not in the connected component of the identity. One example is Witten's global anomaly [42], which destroys the consistency of four-dimensional SU(2) gauge theories with an odd number of left-handed fundamental fermions. As for the role played by anomaly cancellation in string theory, we focused on spacetime (i.e., target space) anomalies, although by no means they are the only type constraining consistent string models. Indeed, the cancellation of the world-sheet conformal anomaly is the crucial element leading to the notion of critical dimension, while modular invariance ensures the absence of global gravitational anomalies on the worldsheet.

我们对反常的概述一直局限于那些破坏量子理论在无穷小微分同胚和规范变换下不变性的情况，这类反常正因如此可在微扰论中被观测到。但反常也可以影响不包含在单位元连通分支中的变换。一个例子是威滕整体反常 [42]，它会破坏存在奇数个左手基本费米子的四维 SU(2) 规范理论的自洽性。至于反常消除在弦论中发挥的作用，我们一直聚焦于时空 (即目标空间) 反常，但它绝不是唯一约束自洽弦模型的反常类型。实际上，世界面共形反常的消除是引出临界维度概念的核心要素，而模不变性保证了世界面上不存在整体引力反常。

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[16] Interestingly, the tadpole term proportional to C_{10} cancels as a consequence of the gauge group being SO(32).

[16] 有趣的是，由于规范群为 SO(32)，正比于 C_{10} 的蝌蚪项会发生抵消。

In page 2244 we learned how perturbative anomalies on an even D -dimensional manifold $\mathcal{Y}$ are obtained from the variation of a $(D+1)$ -dimensional Chern-Simons form, related to the index of a Dirac operator in $D+2$ dimensions. A similar structure exists for global anomalies [43] where the central role is played by the η -invariant, defined as the regularized sum of the signs of the eigenvalues of the Dirac operator

在第 2244 页我们已经了解，偶 D 维流形 $\mathcal{Y}$ 上的微扰反常如何从 $(D+1)$ 维陈-西蒙斯形式的变分得到，该形式和 $D+2$ 维中狄拉克算符的指标相关。整体反常也存在类似结构 [43]，其中核心角色是 η 不变量，它定义为狄拉克算符本征值符号的正则化和。

$$\eta \equiv \left[\sum_{\lambda_i \neq 0} \operatorname{sign}(\lambda_i)\right]_{\text{reg}} = \lim_{\varepsilon \to 0^+} \sum_{\lambda_i \neq 0} \frac{\lambda_i}{|\lambda_i|} e^{-\varepsilon|\lambda_i|}. \tag{130}$$

The variation of the effective action under a global transformation π is given in terms of the η -invariant of the associated mapping torus $\mathcal{T}_\pi \equiv \left(\mathcal{Y} \times S^1\right)_\pi$. This is constructed from the cylinder $\mathcal{Y} \times [0,1]$ by identifying its boundaries modulo the transformation π , as shown on the left of Fig. 4. For a Weyl fermion, the result is

整体变换 π 下有效作用量的变分可以用关联映射环面 $\mathcal{T}_\pi \equiv \left(\mathcal{Y} \times S^1\right)_\pi$ 的 η 不变量表示。映射环面由柱面 $\mathcal{Y} \times [0,1]$ 按变换 π 模等同边界构造而成，如图 4 左侧所示。对于外尔费米子，结果为

$$\Delta_\pi \Gamma_{\text{eff}} \equiv \Gamma_{\text{eff}}^\pi - \Gamma_{\text{eff}} = \frac{i\pi}{2} \eta \mathcal{T}_\pi. \tag{131}$$

The Atiyal-Patodi-Singer (APS) theorem relates this η -invariant to the index of the Dirac operator on a $(D+2)$ -dimensional manifold $\mathcal{B}$ such that $\partial\mathcal{B} = \mathcal{T}_\pi$

阿蒂亚-帕度迪-辛格 (APS) 定理将该 η 不变量与 $(D+2)$ 维流形 $\mathcal{B}$ (满足 $\partial\mathcal{B} = \mathcal{T}_\pi$) 上狄拉克算符的指标联系起来

$$\frac{1}{2}\eta\mathcal{T}_\pi = \mathrm{index}_{\mathcal{B}}(i\mathcal{D}) - \int_{\mathcal{B}}\left[\widehat{A}(\mathcal{R})\,\mathrm{ch}(\mathcal{F})\right]_{D+2}, \tag{132}$$

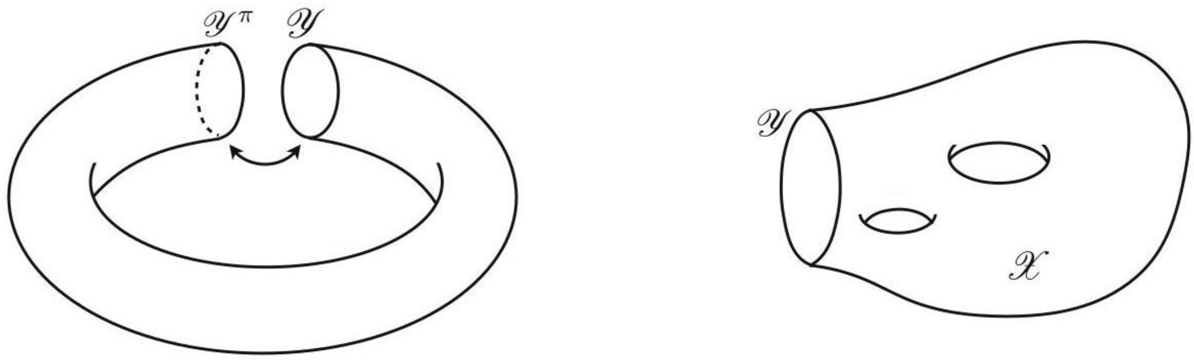

Fig. 4 Left: Construction of the mapping torus associated with a global transformation π . The two boundaries of the cylinder $\mathcal{Y}\times[0,1]$ are glued together modulo π . Right: The Euclidean spacetime $\mathcal{Y}$ is seen as the boundary of a $(D+1)$ -dimensional manifold with all the relevant mathematical structures on $\mathcal{Y}$ properly extended

图 4 左: 整体变换 π 对应的映射环面构造。柱面 $\mathcal{Y}\times[0,1]$ 的两个边界按 π 模粘合在一起。右: 欧几里得时空 $\mathcal{Y}$ 可被视作 $(D+1)$ 维流形的边界，$\mathcal{Y}$ 上所有相关数学结构都被合理延拓

where the second term on the right-hand side vanishes in the absence of perturbative anomalies.

其中不存在微扰反常时，右侧第二项会消失。

The APS η -invariant is in fact the key to a whole new approach to anomalies building on the notion of anomaly inflow and with deep mathematical implications [44-46]. Let us take our Euclidean D -dimensional manifold of interest $\mathcal{Y}$ to be the boundary of some manifold $\mathcal{X}$ (see the right of Fig.4). In order to define fermions properly, we require the spin/pin structure on $\mathcal{Y}$ to be smoothly extended to $\mathcal{X}$. Moreover, boundary conditions for fermions on $\mathcal{Y} = \partial\mathcal{X}$ have to be carefully chosen so the Dirac operator on $\mathcal{X}$ is self-adjoint (see [44-46] for the technical details). Once all mathematical subtleties are properly handled, the partition function for a Weyl fermion on $\mathcal{Y}$ can be written as

APS η 不变量实际上是基于反常流入概念的全新反常研究方法的关键，该方法具有深刻的数学意义 [44-46]。设我们关心的欧几里得 D 维流形 $\mathcal{Y}$ 是某个流形 $\mathcal{X}$ 的边界 (参见图 4 右侧)。为了合理定义费米子，我们要求 $\mathcal{Y}$ 上的自旋/pin 结构能光滑延拓到 $\mathcal{X}$ 。此外，必须谨慎选择 $\mathcal{Y} = \partial\mathcal{X}$ 上费米子的边界条件，以保证 $\mathcal{X}$ 上的狄拉克算符是自伴算符 (技术细节参见 [44-46])。妥善处理所有数学细节后，$\mathcal{Y}$ 上外尔费米子的配分函数可以写为

$$Z \equiv e^{-\Gamma_{\text{eff}}} = |\text{Pf}(i\mathcal{D})| \exp\left(-\frac{i\pi}{2}\eta_{\mathcal{X}}\right) \exp\left(-2\pi i \int_{\mathcal{X}} I^0_{D+1}\right). \tag{133}$$

This expression is independent on changes on either the metric or the gauge field on $\mathcal{X}$ as far as these do not affect their values on $\mathcal{Y}$. This means that, despite appearances, it only depends on field theory data on the boundary.

只要度规或规范场的改变不影响它们在 $\mathcal{Y}$ 上的取值，该表达式就与 $\mathcal{X}$ 上度规或规范场的改变无关。这意味着，尽管形式上看不明显，该表达式仅依赖边界上的场论数据。

The fermion partition function (133) smoothly connects with the standard analysis of perturbative and global anomalies. For the first type, the η -invariant remains unchanged, while the variation of the Chern-Simons form gives the known expression for the integrated anomaly [17] . The partition function may also change under transformations not connected with the identity, and here is where the sewing properties of the η -invariant come in handy. Consider two manifolds $\mathcal{X}_1$ and $\mathcal{X}_2$ being glued together by their boundaries, as shown on the left of Fig. 5. The η -invariant associated with the glued manifold $\mathcal{X}_1 \sqcup \mathcal{X}_2$ is given by

费米子配分函数 (133) 与微扰反常和整体反常的标准分析平滑衔接。对于第一类反常，η 不变量保持不变，而陈-西蒙斯形式的变分给出积分反常 [17] 的已知表达式。配分函数也可能在不与单位元连通的变换下发生变化，而 η 不变量的缝合性质正好可以派上用场。考虑两个流形 $\mathcal{X}_1$ 和 $\mathcal{X}_2$ 沿其边界粘合在一起，如图 5 左侧所示。与粘合后的流形 $\mathcal{X}_1 \sqcup \mathcal{X}_2$ 关联的 η 不变量可表示为

$$e^{-\frac{i\pi}{2}\eta_{\mathcal{X}_1 \sqcup \mathcal{X}_2}} = e^{-\frac{i\pi}{2}\eta_{\mathcal{X}_1}} e^{-\frac{i\pi}{2}\eta_{\mathcal{X}_2}}. \tag{134}$$

We can apply this relation to compute the change in the partition function (133) under a global transformation π , implemented by gluing the corresponding mapping torus to the boundary of $\mathcal{X}$ (see the right of Fig. 5). Using Eq. (134), we find

我们可以利用该关系计算整体变换 π 下配分函数 (133) 的变化，该变换通过将对应的映射环面粘合到 $\mathcal{X}$ 的边界来实现 (参见图 5 右侧)。利用式 (134)，我们得到

$$\frac{Z^\pi}{Z} = e^{-\frac{i\pi}{2}\eta_{\mathcal{T}_\pi}} \tag{135}$$

which reproduces the form of the global anomaly given in (131). In the previous discussion we assumed tacitly that D is even and focused our attention on gauge and gravitational anomalies. For D odd, the Chern-Simons term in the partition function is absent and the standard treatment of parity anomaly in terms of the η -invariant [47] is retrieved.

其重现了 (131) 中给出的整体反常的形式。在之前的讨论中，我们默认假设 D 为偶数，并将注意力集中在规范反常和引力反常上。当 D 为奇数时，配分函数中的陈-西蒙斯项不存在，我们就得到了用 η 不变量表述的宇称反常的标准处理 [47]。

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[17] In Euclidean space the anomaly only affects the imaginary part of the effective action, so $\left|\text{Pf}\left(i\not{D}\right)\right|$ remains invariant.

[17] 在欧氏空间中，反常仅影响有效作用量的虚部，因此 $\left|\mathrm{Pf}\left(i\not{D}\right)\right|$ 保持不变。

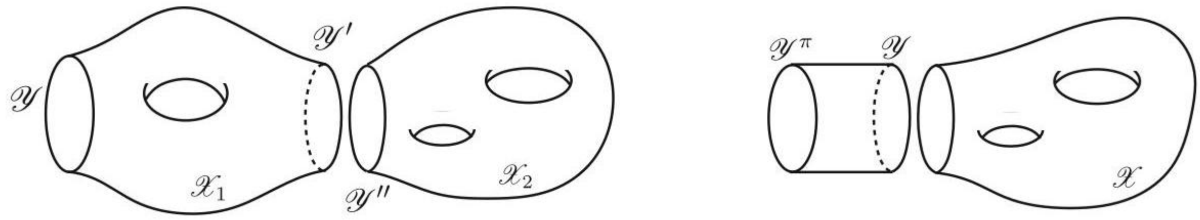

Fig. 5 Left: Gluing two surfaces together along their boundaries $\mathcal{Y}'$ and $\mathcal{Y}''$. Right: Global transformations are implemented by attaching to $\mathcal{Y} = \partial\mathcal{X}$ the appropriate mapping torus

图 5 左: 沿边界 $\mathcal{Y}'$ 和 $\mathcal{Y}''$ 粘合两个曲面。右: 整体变换通过将对应映射环面附加到 $\mathcal{Y} = \partial\mathcal{X}$ 实现

The phase in Eq. (133) can be interpreted as the partition function of an "anomaly" topological field theory defined on $\mathcal{X}$ that includes fermions. An important question is whether there are ambiguities associated with the choice of the manifold $\mathcal{X}$, the so-called Dai-Freed anomalies. In fact, the consistency of the theory on $\mathcal{Y}$ requires then that the fermion partition function does not depend on the higher-dimensional manifold $\mathcal{X}$ used and therefore that it is free from Dai-Freed anomalies. Defining the partition function using two different manifolds $\mathcal{X}_1$ and $\mathcal{X}_2$ with $\mathcal{Y} = \partial\mathcal{X}_1 = \partial\mathcal{X}_2$, this condition takes the form

式 (133) 中的相位可以解释为定义在包含费米子的 $\mathcal{X}$ 上的“反常”拓扑场论的配分函数。一个重要问题是，流形 $\mathcal{X}$ 的选择是否存在歧义，即所谓的戴-弗里德反常。实际上，理论在 $\mathcal{Y}$ 上的自洽性要求费米子配分函数不依赖所选取的高维流形 $\mathcal{X}$ ，因此它不存在戴-弗里德反常。使用两个不同的流形 $\mathcal{X}_1$ 和 $\mathcal{X}_2$ (满足 $\mathcal{Y} = \partial\mathcal{X}_1 = \partial\mathcal{X}_2$) 定义配分函数，该条件可写为

$$1 = \frac{Z_1}{Z_2} = e^{\frac{i\pi}{2}\eta_{\mathcal{X}_1 \sqcup \overline{\mathcal{X}}_2}} \tag{136}$$

where $\mathcal{X}_1 \sqcup \overline{\mathcal{X}}_2$ is the closed manifold resulting from gluing $\mathcal{X}_1$ and $\mathcal{X}_2$ along their common boundary and the bar indicates orientation reversal (the property $\eta_{\overline{\mathcal{X}}} = -\eta_{\mathcal{X}}$ was also used). Notice that $\mathcal{X}_1 \sqcup \overline{\mathcal{X}}_2$ can be seen as the boundary of a $(D+2)$ - dimensional manifold. Using the Stokes theorem together with $dI^0_{D+1} = I_{D+2}$, we find that the integral of the Chern-Simons term over the closed manifold gives an integer resulting in a trivial phase.

其中 $\mathcal{X}_1 \sqcup \overline{\mathcal{X}}_2$ 是将 $\mathcal{X}_1$ 和 $\mathcal{X}_2$ 沿公共边界粘合得到的闭流形，横线表示取向反转 (此处用到了性质 $\eta_{\overline{\mathcal{X}}} = -\eta_{\mathcal{X}}$)。注意，$\mathcal{X}_1 \sqcup \overline{\mathcal{X}}_2$ 可以看作是 $(D+2)$ 维流形的边界。结合斯托克斯定理与 $dI^0_{D+1} = I_{D+2}$ ，我们可知陈-西蒙斯项在闭流形上的积分给出一个整数，对应平庸相位。

Equation (136) means that the cancellation of Dai-Freed anomalies is equivalent to the requirement that the higher-dimensional anomaly topological field theory is trivial. This condition comprises the absence of standard gauge and gravitational (perturbative and global) anomalies, but in fact goes much beyond. It includes a wider class of consistency conditions to be imposed on quantum field theories. This has found interesting application in string theory [48], high-energy phenomenology [46], and the physics of the topological phases of matter [45].

> 式 (136) 表明，戴-弗里德反常的抵消等价于高维反常拓扑场论平庸的要求。该条件包含了标准规范反常和引力反常 (微扰反常与整体反常) 的不存在性，但实际上其适用范围远不止于此。它涵盖了更广一类需要施加在量子场论上的自洽性条件。这已经在弦论 [48]、高能唯象学 [46] 和拓扑物质相物理 [45] 中得到了有趣应用。

But there are more interesting mathematics lurking behind all this. Two closed D -dimensional manifolds $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are in the same equivalence bordism class if their disjoint union $\mathcal{Y}_1 \sqcup \mathcal{Y}_2$ is the boundary of some $(D+1)$ -dimensional manifold. An example of this is illustrated on the left picture of Fig. 5, where from $\mathcal{Y} \sqcup \mathcal{Y}' = \partial\mathcal{X}_1$ we see that $\mathcal{Y}$ and $\overline{\mathcal{Y}}'$ belong to the same bordism class. The discussion above indicates that the cancellation of Dai-Freed anomalies means that consistent fermion theories are invariant under bordisms. In other words, it does not matter what representative $\mathcal{X}$ within a bordism equivalence class we choose to define the theory.

> 但在这一切背后还隐藏着更有趣的数学内容。两个闭合的 D 维流形 $\mathcal{Y}_1$ 和 $\mathcal{Y}_2$ 属于同一个配边等价类，当且仅当它们的不交并 $\mathcal{Y}_1 \sqcup \mathcal{Y}_2$ 是某个 $(D+1)$ 维流形的边界。图 5 的左图展示了一个这样的例子，从中我们可以看出 $\mathcal{Y} \sqcup \mathcal{Y}' = \partial\mathcal{X}_1$ 表明 $\mathcal{Y}$ 和 $\overline{\mathcal{Y}}'$ 属于同一个配边类。上文的讨论表明，戴-弗里德反常的抵消意味着自洽的费米子理论具有配边不变性。换句话说，我们选择配边等价类中的哪一个代表元 $\mathcal{X}$ 来定义理论并不影响结果。

This bordism invariance has led to a fascinating connection with category theory, one of the booming topics in contemporary mathematics (see [49] for a physicist-oriented review). To discuss just the rudiments of category theory and how they apply to the analysis of quantum field theory anomalies lies way beyond the scope of this brief overview. The reader can find a nice introduction to this topic in Ref. [50]. It is in any case interesting how, since their diagrammatic inception in 1969 [51], the subject of quantum field theory anomalies has provided a fertile ground for the application of new mathematics. It was from the late 1970s on [52] that the beautiful and insightful connection with the index theorems emerged. This did not just clarify many features of anomalies already identified in the diagrammatic approach, such as their saturation at one loop, but also highlighted its very general nature independent of the technical details of the Feynman diagram computations where they were first identified. Although the categorial approach to anomalies is still very much under exploration, the expectation exists that it could lead to a deepening of our understanding of quantum fields in some way comparable to what was achieved by the implementation of differential geometry techniques.

这种配边不变性催生了它与范畴论之间极具吸引力的关联，范畴论是当代数学中蓬勃发展的课题之一 (面向物理学家的综述可见文献 [49]。仅讨论范畴论的基础以及它如何应用于量子场论反常分析就远超这篇简短综述的范围，读者可以在文献 [50] 中找到该主题不错的入门介绍。值得一提的是，自 1969 年反常从图论方法中诞生以来 [51]，量子场论反常领域一直是应用新数学的肥沃土壤。直到 20 世纪 70 年代末之后 [52]，它与指标定理之间美妙且富有洞察力的关联才浮现出来。这不仅澄清了在图论方法中已经识别出的反常的诸多性质，比如反常仅在单圈阶存在，还凸显了反常的普适本质——它独立于最初发现反常时所用费曼图计算的技术细节。尽管反常的范畴论方法仍处于探索阶段，但人们期待它能加深我们对量子场的理解，其意义堪比微分几何技术的应用给该领域带来的突破。

Cross-References

交叉引用

- A Lightning Introduction to String Theory

- 弦理论闪电入门

D-Branes

D 膜

D-Branes for Gauge Theories

规范理论用 D 膜

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